Class 11 Properties of solns of the Time independent Schroedinger Eqn

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For $\hat{H} \neq \hat{H}(t)$ the solns to the TDSE:

$$\Psi(\vec{x},t) = e^{-\frac{i}{\hbar}\hat{H}t}\,\Psi(\vec{x},0)$$

also

$$\Psi(\vec{x},0) = \sum_n c_n \psi_n(x) \qquad \hat{H}\psi_n(x) = E_n \psi_n(x)$$

General Properties of the solutions of the TISE:

1) orthogonality: $\int \psi_m^*(\vec{x})\,\psi_n(\vec{x})\,d^3x = \delta_{mn}$

2) Reality: If $\psi_n(x)$ is a soln then so is $\psi_n^*(x)$

3) There is always a lower bound on the allowed energy eigenvalue.

$$\underbrace{E_0}_{\text{lowest allowed energy}} > V_{min}$$

Proof: $\langle \hat{H} \rangle = \int \Psi^*(\vec{x},t)\,\hat{H}\,\Psi(\vec{x},t)\,d^3x$

$$= \int \Psi^*(\vec{x},0)\,\hat{H}\,\Psi(\vec{x},0)\,d^3x$$

If $\Psi(\vec{x},0) \doteq \psi(\vec{x})$ "Superposed state"

$$\langle \hat{H} \rangle = \int \psi^*(\vec{x})\,\hat{H}\,\psi(x)\,d^3x$$

$$= \int \psi^*(\vec{x})\left(\frac{\hat{p}^2}{2m} + V(\vec{x})\right)\psi(x)\,d^3x$$

$$= \frac{1}{2m}\underbrace{\int (\hat{p}\psi(x))^*(\hat{p}\psi(x))\,d^3x}_{\underbrace{\int |\psi'(\vec{x})|^2\,d^3x}_{\text{positive definite}}} + \int \psi^*(\vec{x})\,V(\vec{x})\,\psi(\vec{x})\,d^3x$$

where $\psi'(\vec{x}) = \hat{p}\psi(\vec{x})$

$V(\vec{x})$

$$\langle \hat{H} \rangle \geq \underbrace{\int \psi^*(\vec{x})\,V(\vec{x})\,\psi(\vec{x})\,d^3x}_{\int |\psi(\vec{x})|^2 V(\vec{x})\,d^3x \;\geq\; \int |\psi(\vec{x})|^2 V_{min}\,d^3x}$$

$$\geq V_{min}$$

$$\langle \hat{H} \rangle > V_{min}$$

Recall $\langle \hat{H} \rangle = \sum_n |C_n|^2 E_n \geq V_{min}$ "for Any superposed state"

In the case above $\psi(\vec{x}) = \psi_0(\vec{x})$
↳ the lowest energy state.

$$\Rightarrow \langle \hat{H} \rangle = E_0$$

$$E_0 > V_{min}$$

Example: $V(x) = \frac{1}{2} k x^2 \qquad V_{min} = 0 \Rightarrow E_0 > 0$

4) Parity if $V(-x) = V(x)$ then the solutions can be seperated into even & odd.

5) The solutions to TISE allow for both bound state & un-bound state solutions:

For simplicity we will work in one dimension:

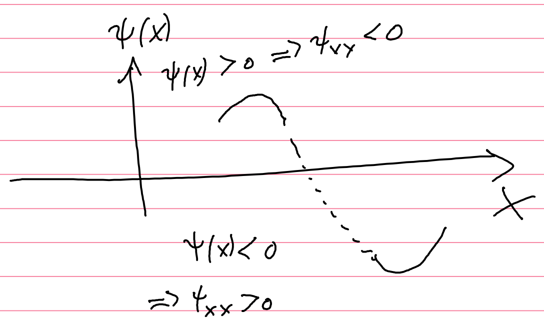
$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\psi_{xx}(x) = \frac{d^2 \psi(x)}{dx^2} = \frac{2m}{\hbar^2} \left(V(x) - E \right) \psi(x)$$

If $E > V(x)$:

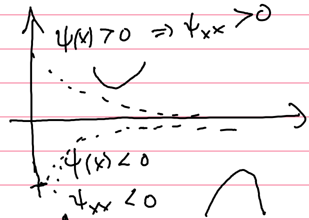
$$\psi_{xx}(x) = -\frac{2m}{\hbar^2} \left(E - V(x) \right) \psi(x)$$

$$= -k^2 \psi(x) \; ; \quad k^2 \geq 0$$

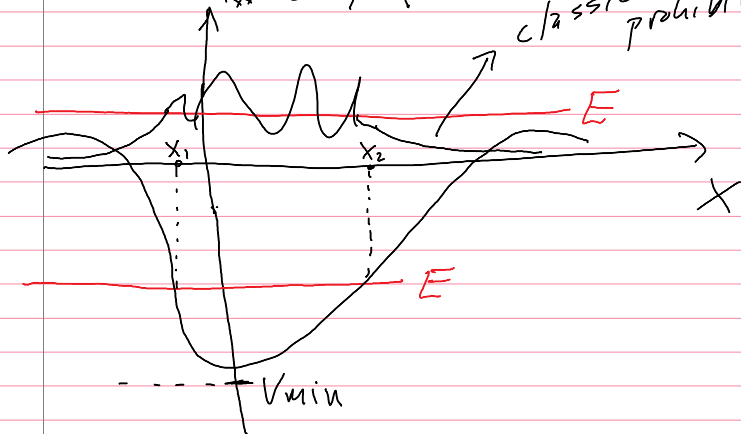


⇒ Solutions oscillate
"Unbound state solutions"

If $E < V(x)$ $\quad \psi_{xx}(x) = k^2 \psi(x) \quad k^2 \geq 0$



"non-oscillatory solution"



$E = T + V$