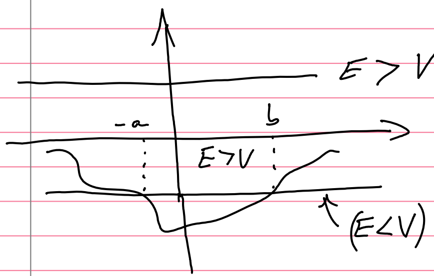
Class 12 The Delta function Potential

Thursday, October 4, 2018 8:02 AM

Solutions of the 1-d time independent Schrodinger Eqn

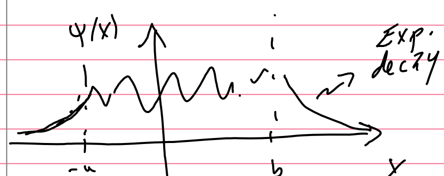
$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \; ; \qquad H\psi(x) = E\psi(x)$$



(No turning points ⇒ unbound solns)

(two turning points ⇒ bound state solns)

when $x > b$ & $x < -a$ classically prohibited regions



"The particle tends to have highest prob. to be found in the region $(-a \le x \le b)$.

Claim: There will always be a bound state soln if there are two turning points, no matter how weak the potential.

Spse $V(x)$ is very weak, for simplicity assume $V(-x) = V(x)$, and it is slowly varying and centered at $x = 0$.

$$\frac{d^2\psi(x)}{dx^2} = +\frac{2m}{\hbar^2}\left(V(x) - E\right)\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = \psi(x)\begin{cases} -k^2 & \text{for } E > V(x) \\ \kappa^2 & \text{for } E < V(x) \end{cases} \qquad \begin{array}{l} k = \sqrt{\frac{2m}{\hbar^2}(E - V(x))} > 0 \\ \kappa = \sqrt{\frac{2m}{\hbar^2}(V(x) - E)} > 0 \end{array}$$

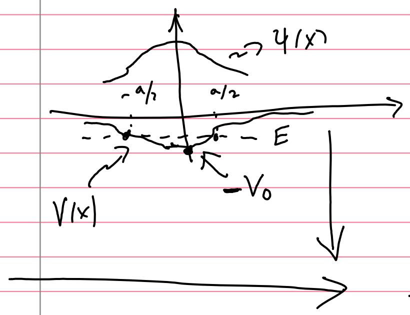
for $E < V(x)$

$$\frac{d^2\psi(x)}{dx^2} = \kappa^2\psi(x) \qquad \kappa = \kappa(x)$$

If $V(x)$ is indep. of x then $\psi(x) \sim e^{\pm \kappa x}$ *

For a general $V(x)$, but very weak & slowly varying $V(x)$ we can assume $\psi(x) \sim e^{-\alpha|x|}$ ↳ trial function

$\hookrightarrow$ trial function



$$\left.\frac{d\psi(x)}{dx}\right|_{x>0} = -\alpha\,\psi(x) \qquad \left.\frac{d\psi(x)}{dx}\right|_{x<0} = \alpha\,\psi(x)$$

$$\int_{-a/2}^{a/2} \frac{d^2\psi(x)}{dx^2}\,dx = \frac{2m}{\hbar^2}\int_{-a/2}^{a/2}\left(V(x)-E\right)\psi(x)\,dx$$

$V(0) = -V_0$

$V'(0) = 0$

$V'(x) = V(x) + V_0$

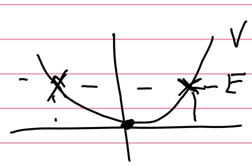
$$\left.\frac{d\psi(x)}{dx}\right|_{x=a/2} - \left.\frac{d\psi(x)}{dx}\right|_{x=-a/2} = \frac{2m}{\hbar^2}\int_{-a/2}^{a/2} V(x)\,\psi(x)\,dx - \frac{2m}{\hbar^2}E\int_{-a/2}^{a/2}\psi(x)\,dx$$

$$\int_{-a/2}^{a/2}\psi(x)\,dx = A\int_{-a/2}^{0} e^{+\alpha x}\,dx + A\int_{0}^{a/2} e^{-\alpha x}\,dx = 0$$

$\psi(\pm a/2)$
$= \psi(a/2)$

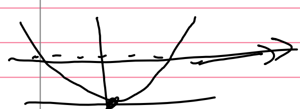
$\left(-|V(x)| - V_0\right)$

$$-2\alpha\,\psi(a/2) = \frac{2m}{\hbar^2}\int_{-a/2}^{a/2} V(x)\,\psi(x)\,dx$$



$\int V(x)\,dx > 0$

$$\approx \frac{2m}{\hbar^2}\,\psi(a/2)\int_{-a/2}^{a/2} V(x)\,dx$$



$$\alpha \approx -\frac{m}{\hbar^2}\int_{-a/2}^{a/2} V(x)\,dx \quad *$$

$V'(x) = V(x) - E$ Note: $\alpha \geq 0 \Rightarrow \int_{-a/2}^{a/2} V(x)\,dx < 0$ in order to have a bound state

$$E \approx -\frac{\hbar^2\alpha^2}{2m}$$ "Bound state energy"

Note: Where we place the origin is arbitrary, we can only measure differences in energy. So what we really found is that,

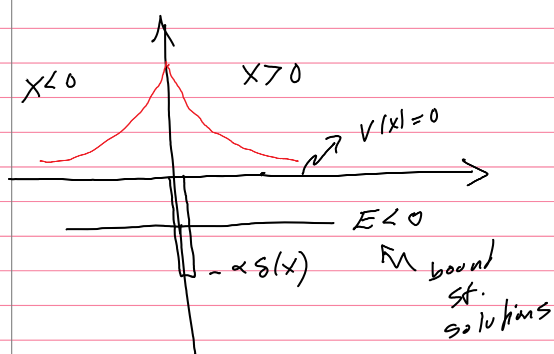
$$\int \left(V(x) - E\right)dx < 0 \quad *$$

Example: The δ-function Potential

This potential is unphysical, of course, but the solutions can provide qualitative information for more complicated potentials w/ the same values for $\int_{-a/2}^{a/2} V(x)\,dx$.

$$V(x) = -\underbrace{\alpha}_{\text{strength}}\,\delta(x) \quad \Rightarrow \quad \int_{-a/2}^{a/2} V(x)\,dx = -\alpha < 0$$

Note if $V(x) = +\alpha\,\delta(x)$ then we should not get a bound state.



$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)}$$

$\underline{x<0}$ $\quad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\,\psi(x)$

$$\psi_{xx}(x) = -\frac{2m}{\hbar^2}E\,\psi(x) \qquad k > 0$$

$$= k^2\,\psi(x) \qquad \boxed{k^2 = -\frac{2m}{\hbar^2}E} > 0$$

$\Rightarrow \psi(x) = A\,e^{\pm kx}$ both for $x<0$ & $x>0$

$$\psi(x) = \begin{cases} A\,e^{kx} & x<0 \\ B\,e^{-kx} & x>0 \end{cases}$$

$$\psi_{x>0}(0) = \psi_{x<0}(0) \Rightarrow A = B$$

$$\psi(x) = A\,e^{-k|x|}$$

If we integrate both sides of the SE:

$$\lim_{a\to 0}\left[-\frac{\hbar^2}{2m}\left[\frac{d\psi(x)}{dx}\right|_{-a/2}^{a/2} - \alpha\,\psi(0)\right] = \lim_{a\to 0} E\int_{-a/2}^{a/2}\psi(x)\,dx = 0$$

$$\lim_{a\to 0}\left.\frac{d\psi(x)}{dx}\right|_{-a/2}^{a/2} = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$$

$$\left.\frac{d\psi}{dx}\right|_{x>0} = -k\,\psi(x) \qquad \left.\frac{d\psi}{dx}\right|_{x<0} = k\,\psi(x)$$

$$\left.\frac{d\psi}{dx}\right|_{x>0} = -\kappa\,\psi(x) \qquad \left.\frac{d\psi}{dx}\right|_{x<0} = \kappa\,\psi(x)$$

$$\lim_{a\to 0}\left(-\kappa\,\psi(a/2) - \kappa\,\psi(-a/2)\right) = \frac{-2m\alpha}{\hbar^2}\,\psi(0)$$

$$-2\kappa\,\psi(0) = -\frac{2m\alpha}{\hbar^2}\,\psi(0)$$

$$\Longrightarrow \boxed{\kappa = \frac{m\alpha}{\hbar^2}} \qquad \text{If } \alpha>0 \Rightarrow \kappa>0 \checkmark$$

$$E = -\frac{\hbar^2\kappa^2}{2m} \Rightarrow \boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$

One bound state soln

"negative" ✓

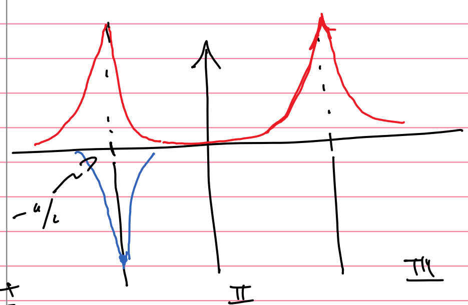
Check the Units! $V(x) = -\alpha\,\delta(x)$

$[E] \qquad [EL]\left[\frac{1}{L}\right]$

$$\int \delta(x)\,dx = 1$$

$1/L \quad L \quad$ 1

$$[E] = \left[\frac{m\alpha^2}{2\hbar^2}\right] = \left[\frac{mc^2\alpha^2}{2(\hbar c)^2}\right] = \frac{[E\,E^2L^2]}{[E\cdot L]^2} = [E] \checkmark$$



$$V(x) = -\alpha\left(\delta(x-a/2) + \delta(x+a/2)\right)$$

$$\psi_+ = \frac{\psi_1(x+a/2) + \psi_1(x-a/2)}{\sqrt{2}} \qquad \text{"even"}$$

$$V(-x) = V(x)$$

$$\psi_- = \frac{\psi_1(x-a/2) - \psi(x+a/2)}{\sqrt{2}} \qquad \text{"odd"}$$

$$E_\pm = ?$$