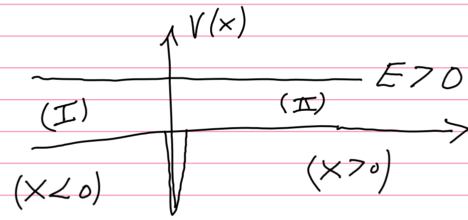
The δ-function Potential

$$V(x) = -\alpha\,\delta(x)$$



I) $x<0$, $x \neq 0$, $V = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) = E\,\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = -k^2\,\psi(x) \qquad k^2 = \frac{2m}{\hbar^2}E \quad E>0$$

Positive definite

$$E = \frac{\hbar^2 k^2}{2m}$$

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

$$\frac{\psi(x)}{A} = e^{ikx} + B/A\, e^{-ikx}$$

$$\psi(x) = e^{ikx} + \tilde{R}\, e^{-ikx} \qquad *$$

"Incoming Wave"

"Reflected wave"

$$\psi(x,t) = e^{-i/\hbar\, \hat{H} t}\,\psi(x,0)$$

$$= e^{i(kx-\omega t)} + S_+ e^{-i(kx+\omega t)}$$

$$\omega = \frac{\hbar k^2}{2m}$$

$$e^{i(k)(\underbrace{x-\omega/k\, t}_{x - vt})}$$

$$v = \omega/k$$

$(x<0)$

$$\psi_{x<0}(x) = e^{ikx} + R\, e^{-ikx}$$

$$e^{-i(kx+\omega t)}$$

$(x>0)$

$$\psi_{x>0} = T e^{ikx} + \underbrace{\cancel{G e^{-ikx}}}_{\to 0}$$

Boundary Conditions:

$$\psi_{x<0}(0) = \psi_{x>0}(0)$$

$$(1+R) = T$$

$$-\frac{\hbar^2}{2m}\left[\frac{d\psi}{dx}\bigg|_{x\to +0} - \frac{d\psi}{dx}\bigg|_{x\to -0}\right] = \alpha\,\psi(0)$$

$$iKT - iK(1-R) = \alpha T$$

$$(1+R) = T$$

$$iK(T-1+R) = \alpha T$$

$$R = \frac{\frac{im\alpha}{\hbar^2}}{K - i\frac{m\alpha}{\hbar^2}} \qquad T = \frac{K}{K - i\frac{m\alpha}{\hbar^2}}$$

$V(x) = -\alpha\,\delta(x)$

$$\beta \cong \frac{m\alpha}{\hbar^2} \Rightarrow \left.\begin{aligned} R &= \frac{i\beta}{K - i\beta} \\ T &= \frac{K}{K-i\beta} \end{aligned}\right\}$$

$$\left.\begin{aligned} |R|^2 &= \frac{\beta^2}{K^2+\beta^2} \\ |T|^2 &= \frac{K^2}{K^2+\beta^2} \end{aligned}\right\} \quad |R|^2 + |T|^2 = 1$$

$|R|^2$: prob. for Reflection

$|T|^2$: prob for transmission

Note: for $V(x) = +\alpha\,\delta(x)$ you get the same expressions for $|R|^2$ & $|T|^2$

We can reproduce the same Results using the current density:

$$J(x,t) = \frac{-i\hbar}{2m}\left[\psi^*(x,t)\frac{d}{dx}\psi(x,t) - \psi(x,t)\frac{d}{dx}\psi^*(x,t)\right]$$

$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$

$\rho = |\psi(\vec{x},t)|^2$

Define, $|R|^2 = \left|\frac{J_R}{\quad}\right|$; $|T|^2 = \left|\frac{J_T}{\quad}\right|$

$\rho = |\psi(\vec{x},t)|^2$ or

Define, $|R|^2 = \left|\frac{J_R}{J_{inc}}\right|$; $|T|^2 = \left|\frac{J_T}{J_{inc}}\right|$

$J_R \doteq J(\psi \to \psi_R)$ $\quad \psi_R(x) = R e^{-ikx}$

$J_T \equiv J(\psi \to \psi_T)$ $\quad \psi_T(x) = T e^{ikx}$

$\psi_{inc} = e^{ikx}$

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Note:

$V(x) = -\alpha\,\delta(x) \Rightarrow V(-x) = V(x)$ "Even"

$[\hat{H}, \hat{\mathbb{P}}]_- = 0 \Rightarrow$ we should be able to find simultaneous eigenfunctions of H & $\mathbb{P}$.

$\therefore$ Lets work with parity eigenstates:

Even parity states

$$\left.\begin{aligned} \psi_{x<0}(x) &= e^{ikx} + S_+ e^{-ikx} \\ \psi_{x>0}(x) &= e^{-ikx} + S_+ e^{+ikx} \end{aligned}\right\} \text{ Then } \psi(-x) = \psi(x)$$

$\psi_{x<0}(-x) = \psi_{x>0}(x)$

For the odd parity states

$$\left\{\begin{aligned} \psi_{x<0}(x) &= e^{ikx} + S_- e^{-ikx} \\ \psi_{x>0}(x) &= -\left(e^{-ikx} + S_- e^{+ikx}\right) \end{aligned}\right. \Rightarrow \psi(-x) = -\psi(x)$$

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B.C. for the Even Parity States

$\psi_{x<0}(0) = \psi_{x>0}(0) \Rightarrow 1 + S_+ = 1 + S_+$ ✓

$$\left.\frac{d\psi}{dx}\right|_{x\to+0} - \left.\frac{d\psi}{dx}\right|_{x\to-0} = -\frac{2m}{\hbar^2}\alpha\,\psi(0)$$

$$\underbrace{-ik[1 - S_+] - ik[1 - S_+]} = -\frac{2m}{\hbar^2}\alpha\,(1 + S_+)$$

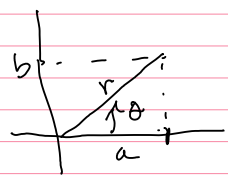
$$+2ik(1-S_+) = \frac{2m}{\hbar^2}\alpha(1+S_+)$$

$$\left\{ \frac{1-S_+}{1+S_+} = -\frac{imα}{\hbar^2 k} = -i\beta/k \right.$$

$$S_+ = \frac{1+i\beta/k}{1-i\beta/k} = \frac{k+i\beta}{k-i\beta}$$

$$\beta \equiv \frac{m\alpha}{\hbar^2}$$

$$S_+ = \frac{k+i\beta}{k-i\beta}; \quad |S_+|^2 = 1$$



S_+ = "A phase factor" $\quad S_+ = e^{2i\delta_+(k)}$

$z = a + ib$

$= re^{i\theta}$

$\tan\theta = b/a$

$$S_+ = \frac{k^2-\beta^2+2i\beta}{k^2+\beta^2} \Rightarrow a = \frac{k^2-\beta^2}{k^2+\beta^2}$$

$$b = \frac{2\beta}{k^2+\beta^2}$$

$$\left\{ 2\delta_+(k) = \tan^{-1}\left(\frac{2\beta}{k^2-\beta^2}\right) \right\}$$

for the even sol'n

For the odd sol'ns $\psi_{x<0}(0) = \psi_{x>0}(0)$

$$\Rightarrow 1+S_- = -(1+S_-)$$

$$\Rightarrow S_- = -1 \qquad 2\delta_- = \pi$$

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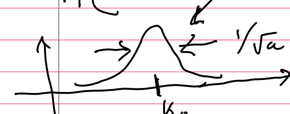
We should really have this discussion w/ wavepackets

for wavepackets:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int dk\, \phi(k)\, e^{i(kx-\omega t)} \qquad \omega = \frac{\hbar k^2}{2m}$$

pick: $\phi(k) = Ae^{-a(k-k_0)^2}$ * $\quad \int|\phi(k)|^2 dk = 1$



$[\phi(k)] \sim [L^{1/2}]$ $\quad 1/L$

From homework you have found that the fourier transform of a gaussian is a gaussian:

$$\psi(x,t) = e^{i(k_0 x - \omega(k_0) t)} \; F(x - v t)$$

$\Delta x \sim t^{1/2}$

$\int |\psi|^2 dx = 1$

$[\psi] \sim \frac{1}{\sqrt{L}}$

$$F(x) \propto \frac{1}{\bar{a}^{1/4}} e^{-\frac{x^2}{4\bar{a}}}$$

$$v = \frac{d\omega}{dk}\Big|_{k=k_0} = \frac{\hbar k_0}{m} = p/m$$

$$\bar{a} = a + i t/2m \quad *$$

$$\psi_{x<0}(x,t) = \frac{1}{\sqrt{2\pi}} \int dk\, \phi(k)\, e^{-i\omega t} \Big[\underbrace{e^{ikx} + S_+ e^{-ikx}} \Big]$$

For the incoming wave:

$$\psi_{\substack{in \\ x<0}}(x,t) = e^{i(k_0 x - \omega(k_0) t)} \; F(x - v_0 t)$$

$$v_0 = \hbar k_0/m$$

For the reflective wave:

$$\psi_{Rf} = e^{-i(k_0 x - \omega(k_0) t + 2i\delta_+(k_0))} \; F(x + v_0 t - 2\delta_+'(k_0))$$

$$\delta_+'(k_0) = \frac{d\delta_+(k)}{dk}\Big|_{k=k_0}$$

The reflected wave will retarded by a time Δt:

$$\Delta t = \frac{2\delta_+'(k_0)}{v_0}$$