Class 14, Delta function Pot, Harmonic Oscillator

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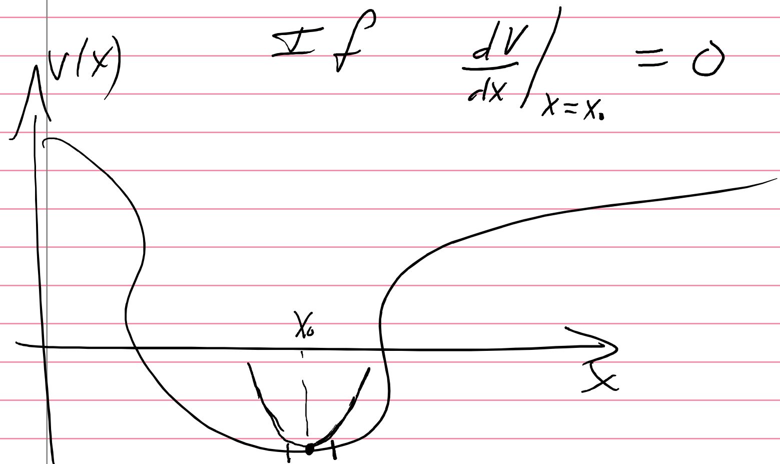
Harmonic Oscillator

For Any Potential $V(x)$:

$$V(x) = V(x_0) + (x-x_0)\frac{dV}{dx}\Big|_{x=x_0} + \frac{1}{2}(x-x_0)^2\frac{d^2V}{dx^2}\Big|_{x=x_0} + \cdots$$



If $\frac{dV}{dx}\Big|_{x=x_0} = 0$

$$V(x) - V(x_0) \approx \frac{1}{2}k(x-x_0)^2$$

"The Harmonic oscillator Potential"

We will discuss the analytic soln to

$$H\psi = E\psi \quad ; \quad V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

$$\omega^2 = k/m$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2}m\omega^2x^2\psi(x) = E\,\psi(x) \quad \cancel{*}$$

only $E > 0$

only bound state solns exist.

"unitless"

1st Step get rid of units! Defn: $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$

$$\left[\frac{m\omega}{\hbar}\right] = \left[\frac{1}{L^2}\right]$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \sqrt{\frac{m\omega}{\hbar}}\frac{\partial}{\partial \xi}$$

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$$\frac{\partial^2}{\partial x^2} = \left(\frac{m\omega}{\hbar}\right)\frac{\partial^2}{\partial \xi^2}$$

$$-\frac{\hbar^{\cancel{2}}}{2\cancel{m}}\left(\frac{\cancel{m}\omega}{\cancel{\hbar}}\right)\frac{\partial^2}{\partial \xi^2}\psi(\xi) + \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\left(\frac{\hbar}{\cancel{m\omega}}\right)\xi^2\psi = E\psi$$

$$-\frac{d^2}{d\xi^2}\psi(\xi) + \xi^2\psi(\xi) = \underbrace{\frac{2E}{\hbar\omega}}_{K}\psi(\xi)$$

$$\boxed{\frac{d^2\psi}{d\xi^2}\psi(\xi) + (\xi^2 - K)\psi(\xi) = 0}$$

$$K = \frac{2E}{\hbar\omega}$$

Asymptotic Soln $\quad \xi \to$ large

$$\frac{d^2\psi(\xi)}{d\xi^2} = -\xi^2\psi(\xi)$$

General soln has the form: $\psi(\xi) = A\xi^n e^{-\xi^2/2} + B\xi^n e^{\xi^2/2}$

$\therefore$ We will search for general solutions of the form:

$$\psi(\xi) = h(\xi)\, e^{-\xi^2/2}$$ where $h(\xi)$ is a polynomial in ξ.

$$\psi'(\xi) = h'(\xi)\, e^{-\xi^2/2} + h(\xi)(-\xi)\, e^{-\xi^2/2}$$

$$h''(\xi) - 2\xi h'(\xi) + (K-1)h(\xi) = 0$$

"Hermite Differential Eqn"

Expand $h(\xi)$ in a Taylor series: $h(\xi) = \sum_{m=0}^{\infty} a_m \xi^m$

Expand $h(\xi)$ in a Taylor series: $\boxed{h(\xi) = \sum_{m=0} a_m \xi^m}$

$$h'(\xi) = \sum_m a_m\, m\, \xi^{m-1}$$

$$h''(\xi) = \sum_m a_m\, m(m-1)\, \xi^{m-2} = \sum_l a_{l+2}\, (l+1)(l+2)\, \xi^l$$

$l = m-2$

$m = l+2$

$m-1 = l+1$

$$= \sum_m a_{m+2}\, (m+1)(m+2)\, \xi^m$$

$$\sum_m \left[a_{m+2}(m+1)(m+2)\xi^m - 2a_m(m)\xi^m + (K-1)a_m \xi^m \right] = 0$$

$$\sum_m \left[a_{m+2}(m+1)(m+2) + (K-1-2m)a_m \right] \xi^m = 0$$

$$\Rightarrow a_{m+2}(m+1)(m+2) + (K-1-2m) = 0$$

$$a_{m+2} = \frac{-(K-1-2m)}{(m+1)(m+2)}\, a_m$$

$$\boxed{a_{m+2} = \frac{2m+1-K}{(m+1)(m+2)}\, a_m}$$

"Recursion Relation"

Note: for $a_0 = 0$, $a_1 \neq 0 \Rightarrow$ all m: odd solns

$a_0 \neq 0$, $a_1 = 0 \Rightarrow$ all m: even solns

We have to check whether the series converges.