Class 15 Harmonic Oscillator cont.

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Analytic Solns of the Harmonic Osc. (cont.)

We have found:
$$\psi(\xi) = h(\xi)\, e^{-\xi^2/2} \quad ; \quad \xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$h(\xi) = \sum_{m=0}^{\infty} a_m \xi^m$$

$$a_{m+2} = \frac{2m+1-K}{(m+1)(m+2)}\, a_m$$

We use the ratio test to check if the series converges:

$$\lim_{m\to\infty} \frac{a_{m+2}}{a_m} = \lim_{m\to\infty} \frac{2m}{m^2} \longrightarrow \lim_{m\to\infty} \underline{\frac{2}{m}} \to 0$$

∴ The series converges.

But we also require $\psi(\xi)$ to normalizable $\Rightarrow \lim_{\xi\to\infty} \psi(\xi) \to 0$

$$\Rightarrow h(\xi)\, e^{-\xi^2/2} \xrightarrow[\xi\to\infty]{} 0$$

∴ $h(\xi)$ cannot grow faster than $e^{\xi^2/2}$ w/ ξ large.

The series for $e^{\xi^2/2}$:

$$e^{\xi^2/2} = \sum_n \frac{1}{n!}\left(\frac{1}{2}\right)^n \xi^{2n}$$

$$= \sum b_n \xi^{2n} \quad ; \quad b_n \equiv \frac{1}{2^n n!}$$

$$\frac{b_{n+1}}{b_n} = \frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = \frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2}\,\frac{1}{n+1} \to \underline{\frac{1}{2n}}$$

∴ We must cut off the series for $h(\xi)$

$$\therefore\ h(\xi) = \sum_{m=0}^{m_{max}} a_m \xi^m = a_0 + a_2\xi^2 + \cdots + a_{m_{max}}\xi^{m_{max}} + 0 + 0$$

$$+ \cancel{a_{max+2}}\, \xi^{m_{max}+2} \to 0$$

$\Rightarrow$ for $m = m_{max}$ $\quad a_{max+2} = 0$

$$a_{(m_{max}+2)} = \frac{2m_{max}+1-K}{(m_{max}+1)(m_{max}+2)}\, a_{max} = 0$$

$$2m_{max} + 1 - K = 0 \quad \Rightarrow \quad 2m_{max} + 1 = \frac{2E}{\hbar\omega}$$

$$E = \hbar\omega\left(m_{max} + \tfrac{1}{2}\right) \qquad 1 - K = -2m_{max} = -2n$$

mutually commuting

$$\{\underbrace{\overbrace{H, \hat{A}, \hat{B}, \hat{C}}}_{}\}$$

"complete set"

↓ eigenvalues

$$\{E, \underbrace{a, b, c}_{q. \#'s}\}$$

Principle Q. #.

Convention: $m_{max} \equiv n$

$$* \quad E_n = \hbar\omega\left(n + \tfrac{1}{2}\right) \quad *$$

$$\therefore \quad a_{m+2} = \frac{2m + 2n}{(m+1)(m+2)}\, a_m = \frac{2(m-n)}{(m+1)(m+2)}\, a_m$$

$$\boxed{a^{(n)}_{m+2} = -\frac{2(n-m)}{(m+1)(m+2)}\, a^{(n)}_m} \qquad n = 2$$

Note: If $m_{max} = n =$ even # the series would not be cut-off for $m =$ odd $\Rightarrow$ either $a_0 = 0$ or $a_1 = 0$

Furthermore, If both a_0 & $a_1 \neq 0$ then can have,

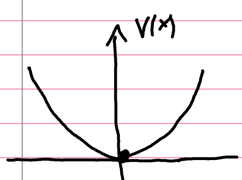
$$\psi(x) = \psi_e(x) + \psi_o(x)$$

but then $H\psi(x) = \hbar\omega(n_{even} + \tfrac{1}{2})\,\psi_e + \hbar\omega(n_{odd} + \tfrac{1}{2})\,\psi_o(x)$

although $[H, \mathbb{P}] = 0$ there is no degeneracy because,

$$\mathbb{P}\,\psi(x) \neq \psi'(x)$$

The smallest value of n is $n = 0$: $\Rightarrow m = 0$



$$h(\xi) = a_0^{(0)} \quad \Rightarrow \quad \psi_0(\xi) = a_0\, e^{-\xi^2/2}$$

"Gnd state wavefunction"

$$E_0 = \frac{\hbar\omega}{2} \neq 0$$

"Zero Point Energy"

Normalization: $\int |\psi_0(x)|^2\, dx = 1 \quad \Rightarrow \quad \int_{-\infty}^{\infty} |\psi_0(\xi)|^2\, d\xi = \sqrt{\frac{m\omega}{\hbar}}$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$dx = \sqrt{\frac{\hbar}{m\omega}}\, d\xi$$

$$|a_0|^2 \int_{-\infty}^{\infty} e^{-\xi^2}\, d\xi = \sqrt{\frac{m\omega}{\hbar}} = |a_0|^2 \sqrt{\pi} = \sqrt{\frac{m\omega}{\hbar}}$$

$$|a_0|^2 \int_{-\infty} e^{-3}\, d\xi = \sqrt{\frac{m\omega}{\hbar}} \quad = |a_0|^2 \sqrt{\pi} = \sqrt{\frac{m\omega}{\hbar}}$$

$$a_0^{(0)} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sim \left[\frac{1}{L^{1/2}}\right] \qquad \frac{mc^2}{\pi\hbar c}\frac{\omega}{c} = \frac{E}{E\cdot L}\frac{1}{L} \sim \frac{1}{L^2}$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega}{\hbar}\right)X^2/2}$$

$\therefore$ for the gnd state: $(\Delta X)_0 (\Delta P)_0 = \frac{\hbar}{2}$

i.e. the uncertainty product is the smallest it can be.

For the even solutions : $a_1^{(n)} = 0 \Rightarrow a_{2n+1} = 0$

$n=2$

$$a_2^{(2)} = \frac{-2(2-0)}{(0+1)(0+2)} a_0^{(2)} = -\frac{4}{2} a_0^{(2)} = -2a_0^{(2)}$$

$$h_n(\xi) = \sum_{m=0}^{n} a_m^{(n)} \xi^m \overset{n=2}{=} a_0^{(2)} + a_2^{(2)} \xi^2$$

$$h_2(\xi) = a_0^{(2)}(1 - 2\xi^2)$$

$$\psi_2(\xi) = a_0^{(2)}(1-2\xi^2)\, e^{-\xi^2/2} \qquad a_0^{(2)} = \left(\frac{m\omega}{2\pi\hbar}\right)^{1/4}$$

$$E_2 = \frac{3}{2}\hbar\omega$$

for $n=4$:

$$\psi_4(\xi) = a_0^{(4)}\left(1 - 4\xi^2 + \tfrac{4}{3}\xi^4\right) e^{-\xi^2/2}$$

$$E_4 = \tfrac{9}{2}\hbar\omega$$

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Operator Approach to the Harmonic Oscillator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dX^2} + \frac{1}{2}m\omega^2 X^2 \quad ; \quad H\psi(x) = E\psi(x)$$

$$\hat{P} = -i\hbar\frac{\partial}{\partial x} \quad ; \quad \hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

$$\hat{H} = \frac{m\omega^2}{\ } \left(\frac{\hat{P}^2}{\ } + X^2\right) = m\omega^2 \left(i\hat{P} + \hat{X}\right)\left(-i\hat{P} + \hat{X}\right)$$

$$\hat H = \frac{m\omega^2}{2}\left(\frac{\hat p^2}{m^2\omega^2} + \hat X^2\right) = \frac{m\omega^2}{2}\left(\frac{i\hat p}{m\omega} + \hat x\right)\left(\frac{-i\hat p}{m\omega} + \hat x\right)$$

$$+ \frac{i m\omega^2}{2}\left(\frac{\hat x\hat p - \hat p\hat x}{m\omega}\right)$$

$$\hat H = \frac{m\omega^2}{2}\left(\hat x + \frac{i\hat p}{m\omega}\right)\left(\hat x - \frac{i\hat p}{m\omega}\right) - \frac{\hbar\omega}{2}$$

$$\frac{\hat H}{\hbar\omega} = \frac{m\omega}{2\hbar}\left(\hat x + \frac{i\hat p}{m\omega}\right)\left(\hat x - \frac{i\hat p}{m\omega}\right) - 1/2$$

$$= \frac{1}{\sqrt2}\sqrt{\frac{m\omega}{\hbar}}\left(\hat x + i\hat p/m\omega\right)\frac{1}{\sqrt2}\sqrt{\frac{m\omega}{\hbar}}\left(\hat x - \frac{i\hat p}{m\omega}\right) - 1/2$$

Note: $\sqrt{\frac{m\omega}{\hbar}}\,\hat x = \xi$; $\sqrt{\frac{m\omega}{\hbar}}\,\frac{i\hat p}{m\omega} = \sqrt{\frac{m\omega}{\hbar}}\,\frac{\hbar}{m\omega}\frac{d}{dx} = \frac{\partial}{\partial\xi}$

$$\frac{\hat H}{\hbar\omega} = \underbrace{\frac{1}{\sqrt2}\left(\hat\xi + \frac{\partial}{\partial\xi}\right)}_{\hat a}\underbrace{\frac{1}{\sqrt2}\left(\hat\xi - \frac{\partial}{\partial\xi}\right)}_{\hat a^+} - 1/2 \quad ; \quad \xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

Note Claim: $\left(\frac{\partial}{\partial\xi}\right)^+ = -\frac{\partial}{\partial\xi}$

Proof $\int \psi^*(\xi)\frac{\partial}{\partial\xi}\psi(\xi)\,d\xi = -\int\left(\frac{\partial}{\partial\xi}\psi(\xi)\right)^*\psi(\xi)\,d\xi + \cancel{|\psi(x)|^2\Big|_{-L}^{L}} \to 0$

$$\hat H = \hbar\omega\left(\hat a\hat a^+ - 1/2\right)$$

$$[\hat a, \hat a^+] = 1$$

$$[\hat x, \hat p] = i\hbar$$

$$[a, a^+]\,f(\xi) = \frac12\left(\xi + \frac{\partial}{\partial\xi}\right)\left(\xi - \frac{\partial}{\partial\xi}\right)f(\xi) - \frac12\left(\xi - \frac{\partial}{\partial\xi}\right)\left(\xi + \frac{\partial}{\partial\xi}\right)f(\xi) = 1\ f(\xi)$$

$$\hat a \equiv \frac{1}{\sqrt2}\left(\xi + \frac{\partial}{\partial\xi}\right) = \frac{1}{\sqrt2}\sqrt{\frac{m\omega}{\hbar}}\left(\hat x + \frac{i\hat p}{m\omega}\right)$$

$$\hat a^+ \equiv \frac{1}{\sqrt2}\left(\xi - \frac{\partial}{\partial\xi}\right) = \frac{1}{\sqrt2}\sqrt{\frac{m\omega}{\hbar}}\left(\hat x - \frac{i\hat p}{m\omega}\right)$$

$$\boxed{\hat x = \sqrt{\frac{\hbar}{m\omega}}\left(\frac{\hat a + \hat a^+}{\sqrt2}\right) \qquad \hat p = -i\sqrt{\hbar m\omega}\left(\frac{\hat a - \hat a^+}{\sqrt2}\right)}$$

$$\hat a\hat a^+ - \hat a^+\hat a = 1 \Rightarrow \hat a\hat a^+ = 1 + \hat a^+\hat a$$

$$\Rightarrow \boxed{\hat H = \hbar\omega\left(\hat a^+\hat a + 1/2\right)}$$

$$= \hbar\omega\left(\hat N + 1/2\right) \quad \text{w/} \quad \hat N = \hat a^+\hat a$$

$\hat{N}^+ = \hat{a}^+ \hat{a}$ "Hermitian" !

We want to solve:

$$\hat{H}\psi(x) = E\psi(x)$$