Class 16 Harmonic Oscillator Operator Method

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$$\hat{H}\psi(x) = E\psi(x)$$

$$\hat{a} \equiv \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial\xi}\right)$$

$$\hat{H} = \hbar\omega\left(\hat{N} + 1/2\right)$$

$$\hat{a}^+ \equiv \frac{1}{\sqrt{2}}\left(\xi - \frac{\partial}{\partial\xi}\right)$$

$$\hat{N} \equiv \hat{a}^+\hat{a} \quad ; \quad [\hat{a}, \hat{a}^+] = 1$$

Number operator : $\hat{N}^+ = \hat{N}$

$$\hat{a} \xrightarrow{\mathbb{P}} -\hat{a}$$

$$\hat{a}^+ \xrightarrow{\mathbb{P}} -\hat{a}^+$$

claim: $[\hat{N}, \hat{a}] = -\hat{a} \quad ; \quad [\hat{N}, \hat{a}^+] = \hat{a}^+$

Proof $[\hat{a}^+\hat{a}, \hat{a}] = \hat{a}^+\cancelto{0}{[\hat{a}, \hat{a}]} + [\hat{a}^+, \hat{a}]\hat{a} = -\hat{a}$ Q.E.D.

We need the eigenfunctions of $\hat{N}$. Since $\hat{N}$ is Hermitian its eigenvalues must be real. Spse $\psi(x)$ is an Eigenfunction of $\hat{N}$:

$$\hat{N}\psi(x) = n\,\psi(x)$$

$\hookrightarrow$ Real Number!

Now consider, $\hat{N}\left(\hat{a}\psi(x)\right) = \left(-\hat{a} + \hat{a}\hat{N}\right)\psi(x)$
$= (n-1)\,\hat{a}\psi(x)$

$$[\hat{N}, \hat{a}] = -\hat{a}$$
$$\hat{N}\hat{a} - \hat{a}\hat{N} = -\hat{a}$$
$$(\hat{N}\hat{a} = -\hat{a} + \hat{a}\hat{N})$$

$\Rightarrow$ $\hat{a}\psi(x)$ is an eigenfunction of $\hat{N}$ w/ eigenvalue $(n-1)$

It follows that $\hat{a}^2\psi(x)$ is an eigenfunction of $\hat{N}$ with eigenvalue $(n-2)$.

Likewise, $\hat{N}\hat{a}^+\psi(x) = (n+1)\,\hat{a}^+\psi(x)$

$\Rightarrow$ $\hat{a}^+\psi(x)$ is an eigenfunction of $\hat{N}$ w/ eigenvalue $(n+1)$.

Summarizing: given that $\hat{N}\psi(x) = n\,\psi(x)$ *

then $\hat{a}^m\psi(x)$ will an eigenfunction of $\hat{N}$ w/ eigenvalue $(n-m)$

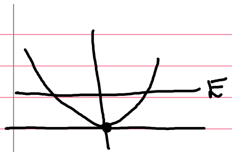
$(\hat{a}^+)^m\psi(x)$ will be an eigenfunction of $\hat{N}$ w/ eigenvalue $(n+m)$

Note: Under Parity:

$$\hat{a}^m\psi(x) \xrightarrow{\mathbb{P}} (-1)^m\,\hat{a}^m\,\psi(-x)$$

$$(\hat{a}^+)^m\psi(x) \xrightarrow{\mathbb{P}} (-1)^m\,\hat{a}^{+m}\,\psi(-x)$$

The eigenenergies corresponding to $\hat{a}^m\psi(x)$ are $E = \hbar\omega\left(n - m + 1/2\right) > 0$



Since $E \geq 0 \Rightarrow$ that there is a limit to how large m can be: $m < n + 1/2$

n has to be positive definite:

$$\hat{N}\psi_n(x) = n\,\psi_n(x) \qquad \langle \hat{H} \rangle_n \geq 0$$

$$
\left.
\begin{aligned}
\langle \hat{H} \rangle_n &= \int dx\, \psi_n^*(x)\, \hat{H}\, \psi_n(x) \\
&= \hbar\omega \int dx\, \psi_n^*(x) \left(\hat{N} + 1/2\right) \psi_n \\
&= \hbar\omega \left(n + 1/2\right) \int dx\, |\psi_n(x)|^2 \\
&= \hbar\omega \left(n + 1/2\right) \geq 0 \quad \Longrightarrow \quad n \geq -1/2
\end{aligned}
\right\}
$$

But n has to be positive!

$$\langle \hat{N} \rangle_n = n = \int dx\, \psi_n^\dagger(x)\, \hat{a}^\dagger \hat{a}\, \psi_n(x) = \int dx \left(\hat{a}\psi_n(x)\right)^\dagger \left(\hat{a}\psi_n(x)\right)$$

$$= \int dx\, \psi'^*(x)\, \psi'(x) = \int dx\, |\psi'(x)|^2 \geq 0$$

$$\Longrightarrow n \geq 0$$

$\therefore$ the smallest value of n is <u>zero</u>.

$\therefore \exists\ \psi_0(x) \ni$:

$$\hat{N}\psi_0(x) = 0\,\psi_0(x) = 0$$

$\Rightarrow \hat{a}^\dagger \hat{a}\, \psi_0(x) = 0$ this will be satisfied if $\hat{a}\,\psi_0(x) = 0$!

We can use this condition to find $\psi_0(x)$ and then generate an infinite # of solutions by using $\psi_n \propto (a^+)^n \psi_0(x)$

$$\psi_n(x) = C_n (a^+)^n \psi_0(x) \;; \qquad \psi_{n+1}(x) = A_n \hat{a}^+ \psi_n(x)$$

$$\psi_{n-1}(x) = A_n' \,\hat{a}\, \psi_n(x)$$

A & A' can be set such that if ψ_n is normalized ψ_{n+1} & ψ_{n-1} will be normalized.

Spse $\int dx\, |\psi_n(x)|^2 = 1 \;; \quad \int dx\, |\psi_{n+1}(x)|^2 = |A|^2 \int dx \left(\hat{a}^+\psi_n(x)\right)^* \left(\hat{a}^+\psi_n(x)\right)$

$$\int dx\, |\psi_{n+1}(x)|^2 = |A|^2 \int dx\, \psi_n^*(x) \underbrace{\hat{a}\,\hat{a}^+}_{\hat{a}^+\hat{a}+1} \psi_n(x)$$

By defn of Hermitian Conjugate

$$= |A|^2 \int dx\, \psi_n^*(x) \left(\hat{a}^+\hat{a}+1\right) \psi_n(x)$$

$$= |A|^2 (n+1) \int dx\, |\psi_n(x)|^2 = |A|^2 (n+1)$$

$$= 1 \Rightarrow |A| = \frac{1}{\sqrt{n+1}} \Rightarrow \psi_{n+1}(x) = \frac{\hat{a}^+ \psi_n(x)}{\sqrt{n+1}}$$

Likewise, $\psi_{n-1}(x) = \dfrac{\hat{a}\,\psi_n(x)}{\sqrt{n}}$

$$\psi_1(x) = \frac{\hat{a}^+ \psi_0(x)}{\sqrt{1}} \;;\; \psi_2(x) = \frac{\hat{a}^+ \psi_1(x)}{\sqrt{2}} \;;\; \psi_3(x) = \frac{\hat{a}^+ \psi_2(x)}{\sqrt{3}}$$

$$\psi_2(x) = \frac{(\hat{a}^+)^2 \psi_0(x)}{\sqrt{1\cdot 2}} \;;\; \psi_3(x) = \frac{(\hat{a}^+)^3}{\sqrt{1\cdot 2\cdot 3}} \psi_0(x)$$

$$\vdots$$

$$\boxed{\psi_n(x) = \frac{(\hat{a}^+)^n}{\sqrt{n!}} \psi_0(x)} \qquad c_n = \frac{1}{\sqrt{n!}}$$

this will give $\psi_n(x)$ normalized if $\psi_0(x)$ is normalized.

To find $\psi_0(x)$ we use:

$$\hat{a}\,\psi_0(x) = 0 \qquad \hat{a} = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial \xi}\right)$$

$$\left(\xi + \frac{\partial}{\partial \xi}\right)\psi_0(\xi) = 0 \qquad \frac{d\psi_0(\xi)}{d\xi} = -\xi\,\psi_0(\xi)$$

$$\frac{d\psi_0(\xi)}{\psi_0(\xi)} = -\xi\, d\xi \Rightarrow \ln \psi_0(\xi) = -\frac{\xi^2}{2} + c$$

$$\psi_0(\xi) = c' e^{-\xi^2/2}$$

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2}} \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x$$

$$\psi_0(x) = A_0\, e^{-\xi^2/2} \qquad A_0 \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\psi_n(x) = A_n (\hat{a}^+)^n e^{-\xi^2/2} \qquad A_n = \frac{A_0}{\sqrt{n!}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{n!}}$$

$$\psi_n(x) = \frac{A_0}{2^{n/2}\sqrt{n!}} \underbrace{\left(\xi - \frac{\partial}{\partial\xi}\right)^n e^{-\xi^2/2}}$$

$$\psi_n(x) = \frac{A_0}{2^{n/2}\sqrt{n!}} H_n(\xi)\, e^{-\xi^2/2} \qquad H_n(\xi) = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial\xi^n} e^{-\xi^2}$$

↳ "Polynomial in ξ of order n"

$$E_n = \hbar\omega(n+1/2)$$

Claim: The set of a $\psi_n(x)$ is complete (needs to be proved)

i.e. Any arbitrary function $f(x)$ which is normalizable can be expressed,

$$f(x) = \sum_n c_n \psi_n(x)$$

$$\int_{-\infty}^{\infty} H_n(\xi) H_m(\xi) e^{-\xi^2} d\xi = \delta_{nm} \pi^{1/2} 2^n n!$$

Also, recall that:

$$\int dx\, \psi_n^*(x)\, \psi_m(x) = \delta_{mn}$$

"Orthogonality"

Note that $\psi_n(-x) = (-1)^n \psi_n(x)$

$$H_n(-x) = (-1)^n H_n(x)$$

Also we know, $$\sum_n \psi_n^*(x)\, \psi_n(x') = \delta(x-x')$$

↳ "Completeness"
"Closure"

Side note: $\hat{N} = a^+ a$ "Differential operator"

$$\hat{N}\psi_n = n\psi_n \qquad n = 0, 1, \ldots$$

If we know all ψ_n & n then we know everything we need to know about $\hat{N}$. ∴ $\hat{N}$ can of as a matrix in the space spanned by the set of eigenfunctions ψ_n:

Defn the matrix N_{mn}

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$$N_{mn} = \int \psi_m^*(x)\, \hat{N}\, \psi_n(x)\, dx$$

$$= n\, \delta_{mn}$$

$$\underset{\sim}{N} = \begin{pmatrix} 0 & & & \phi \\ & 1 & & \\ & & 2 & \\ \phi & & & 3 \ddots \end{pmatrix}$$

An infinite dimensional matrix w/ the eigenvalues along the diagonal.

It is said that $\hat{N}$ is diagonal in the ψ_n basis.

Likewise, for the operators $\hat{a}$ & $\hat{a}^\dagger$:

$$(\underset{\sim}{a})_{mn} = \int \psi_m(x)\, \hat{a}\, \psi_n(x)\, dx \qquad \hat{a}\,\psi_n(x) = \sqrt{n}\, \psi_{n-1}(x)$$

$$= \sqrt{n}\; \delta_{(m,\, n-1)}$$

$$\underset{\sim}{a} = \begin{array}{c|c} & n = 0 \quad 1 \quad 2 \quad \cdots \\ \hline \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \end{array} & \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ \phi & & 0 & \sqrt{3} \\ & & & 0 \ddots \end{pmatrix} \end{array} ; \qquad \underset{\sim}{a}^\dagger = \begin{pmatrix} 0 & & & \phi \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & 0 & \\ & & \ddots & \ddots \end{pmatrix}$$