Class 18 Coherent States

Tuesday, October 30, 2018 8:02 AM

We have seen that the classical limit of the S.S. solutions for the H.O. was obtained by taking the limit n very large

$$E_n = \hbar\omega(n + 1/2) \qquad E_{n+1} - E_n = \hbar\omega <<< E_n$$

$$\lim_{n\to\infty} |\psi_n(x)|^2 \sim \rho_{class}(x) \Big\}$$ Not very complete description of the classical limit. Rather it is a superposition of the ψ_n's in the limit of large n which gives ρ_{class}.

Recall for S.S. of the H.O.

$$\Delta X \Delta P = \hbar(n + 1/2) > \hbar/2$$

This uncertainty product gets large as n gets large, whereas we would expect it to get small in the Classical Limit.

So how do we really get the Classical Limit of the H.O.?

Furthermore for the EM field we will see $(K_0 t - \vec{K}\cdot\vec{x})$

$$\hat{\vec{\mathcal{E}}} = \sum_{K,\lambda} i k\, \epsilon_\lambda(K) \left\{ \hat{a}_\lambda(K)\, e^{iK\cdot x} - \hat{a}^\dagger_\lambda(K)\, e^{-iK\cdot x} \right\}$$

"electric field operator"

$$\hat{H} = \sum_{K,\lambda} \hbar\omega \left(a^\dagger_\lambda(K)\, a_\lambda(K) + 1/2 \right)$$

$$\langle \hat{\vec{\mathcal{E}}} \rangle = 0 \qquad a|n\rangle \sim |n-1\rangle \quad \int \psi_n^* a \psi_n\, dx$$

$$\langle n|a|n\rangle = \langle n|a^\dagger|n\rangle = 0$$

$\therefore$ The Physical States describing the Classical States of the H.O. (or similarly the EM field states) must be a linear superposition of the $\psi_n(x)$ $\ni$ $\Delta X \Delta P = \hbar/2$

In Classical Mechanics of the H.O.

$$\boxed{\begin{aligned} X(t) &= X(0)\sin\omega t \\ P(t) &= m X(0)\,\omega \cos\omega t \end{aligned}} \qquad \omega = \sqrt{\frac{k}{m}}$$

In Q.M. $\langle \hat{X} \rangle_t$ & $\langle \hat{P} \rangle_t$

$\{ \hat{\ } \quad [\,(\hat{a}, \hat{a}^\dagger)$

$$\boxed{\begin{aligned}\hat{X} &= \sqrt{\frac{\hbar}{m\omega}}\left(\frac{\hat{a}+\hat{a}^{+}}{\sqrt{2}}\right)\\ \hat{P} &= -i\sqrt{\hbar m\omega}\left(\frac{\hat{a}-\hat{a}^{+}}{\sqrt{2}}\right)\end{aligned}}$$

Operators (time independent)

$\langle \hat{X}\rangle_t$ we need $\langle \hat{a}\rangle_t$, $\langle \hat{a}^{+}\rangle_t$

$$\frac{d\langle \hat{a}\rangle_t}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{a}]\rangle_t$$

$[a^{+}a, a] = a^{+}[a,a] + [a^{+},a]a = -a$

$$= -\frac{i}{\hbar}(\hbar\omega)\langle \hat{a}\rangle = -i\omega\langle \hat{a}\rangle$$

$$\boxed{\langle \hat{a}\rangle_t = \langle \hat{a}\rangle_{t=0}\, e^{-i\omega t}}$$

$$\frac{d}{dt}\langle \hat{a}^{+}\rangle_t = \frac{i}{\hbar}\langle [\hat{H},\hat{a}^{+}]\rangle = \frac{i}{\hbar}\hbar\omega\,\langle \hat{a}^{+}\rangle$$

$$\boxed{\langle \hat{a}^{+}\rangle_t = \langle \hat{a}^{+}\rangle_0\, e^{i\omega t}}$$

$\langle \hat{X}\rangle_0 = \sqrt{\frac{\hbar}{m\omega}}(\alpha + \alpha^{*})$

$\langle \hat{P}\rangle_0 = -i\sqrt{\hbar m\omega}(\alpha - \alpha^{*})$

$$\langle \hat{X}\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a}\rangle_0 e^{-i\omega t} + \langle a^{+}\rangle_0 e^{i\omega t}\right)$$

$$\langle \hat{P}\rangle_t = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle \hat{a}\rangle_0 e^{-i\omega t} - \langle \hat{a}^{+}\rangle_0 e^{i\omega t}\right)$$

$$\langle \hat{X}\rangle_t = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a}\rangle_0 + \langle \hat{a}^{+}\rangle_0\right)\cos\omega t$$

$\sqrt{\frac{\hbar}{m\omega}}\frac{\sqrt{\hbar m\omega}}{\sqrt{\hbar m\omega}} = \frac{1}{m\omega}\sqrt{\hbar m\omega}$

$$-i\sqrt{\frac{\hbar}{2m\omega}}\left(\langle \hat{a}\rangle_0 - \langle \hat{a}^{+}\rangle_0\right)\sin\omega t$$

$$\langle \hat{X}\rangle_t = \langle \hat{X}\rangle_0\cos\omega t + \frac{\langle \hat{P}\rangle_0}{m\omega}\sin\omega t$$

$$\langle \hat{P}\rangle_t = -i\sqrt{\hbar m\omega}\left(\langle \hat{a}\rangle_0 - \langle \hat{a}^{+}\rangle_0\right)\cos\omega t - i\sqrt{\hbar m\omega}\left(\langle \hat{a}\rangle_0 + \langle \hat{a}\rangle_0\right)\sin\omega t$$

$$\langle \hat{P}\rangle_t = \langle \hat{P}\rangle_0\cos\omega t - \sqrt{\hbar m\omega}\left(\sqrt{\frac{m\omega}{\hbar}}\right)\langle \hat{X}\rangle_0\sin\omega t$$

$$\langle \hat{X}\rangle_t = \langle \hat{X}\rangle_0\cos\omega t + \frac{\langle \hat{P}\rangle_0}{m\omega}\sin\omega t$$

$$\boxed{\begin{aligned} \langle \hat{x} \rangle_t &= \langle \hat{x} \rangle_0 \cos\omega t + \frac{\langle \hat{p} \rangle_0}{m\omega} \sin\omega t \\ \langle \hat{p} \rangle_t &= -m\omega \langle \hat{x} \rangle_0 \sin\omega t + \langle \hat{p} \rangle_0 \cos\omega t \end{aligned}}$$

But for stationary states: $\langle \hat{x} \rangle_0 = 0$!

$$\langle \hat{x} \rangle_0 \propto \langle n | \hat{a} + \hat{a}^\dagger | n \rangle \propto \sqrt{n} \langle n | n-1 \rangle + \sqrt{n+1} \langle n | n+1 \rangle$$

$$\langle \hat{p} \rangle_0 \propto \langle n | \hat{a} - \hat{a}^\dagger | n \rangle = 0$$

$\therefore$ To get the classical limit we have to work with superposed states that saturate the minimum uncertainty product. Such states are called <u>Coherent States</u>

Recall that for the gnd state, $n=0$, $\Delta x \Delta p = \hbar/2$

$$\hat{a}\, \psi_0(x) = 0\, \psi_0(x) = 0 \qquad \psi_0(x) \sim e^{-\xi^2/2} \quad \xi = \sqrt{\tfrac{m\omega}{\hbar}}\, x$$
$$\Delta x \Delta p = \hbar/2$$

Note $\psi_0(x)$ is an eigenstate of $\hat{a}$ with eigenvalue zero.

Claim: the eigenstates of $\hat{a}$ are the coherent states that result in a minimum uncertainty product and which will give the proper classical behavior in the limit of large n.

We want to find the eigenstates of $\hat{a}$ $\ni$ $\hat{a}\,\psi_\alpha(x) = \alpha\, \psi_\alpha(x)$

Note that since $\hat{a}$ is not Hermitian then α need not be real & the eigenstates need not be orthogonal.

Define the Shift Operator:

$$D_\alpha \equiv e^{(\alpha \hat{a}^\dagger - \alpha^* \hat{a})} \quad , \quad D_\alpha^\dagger = e^{-(\alpha a^\dagger - \alpha^* a)} = D_{-\alpha} = D_\alpha^{-1}$$

"Unitary operator"

Claim: $\boxed{D_\alpha^\dagger\, \hat{a}\, D_\alpha = \hat{a} + \alpha \mathbb{1}}$ * $\Rightarrow [a, D_\alpha] = \alpha$

$$D_\alpha \; a \; D_\alpha = a + \alpha \quad \Rightarrow [a, D_\alpha] = \alpha$$

"Unitary operator"

$\therefore$ $\hat{a}\psi_0(x) = 0 \; \psi_0(x)$

$$D_\alpha \psi_0(x) = ?$$

$$\hat{a}\left(D_\alpha \psi_0(x)\right) = D_\alpha \left(\hat{a} + \alpha\right)\psi_0(x)$$
$$= \alpha\left(D_\alpha \psi_0(x)\right)$$

$$\left(\hat{a} D_\alpha = D_\alpha \left(\hat{a} + \alpha\right)\right)$$

$$\therefore \quad D_\alpha \psi_0(x) = N\psi_\alpha(x) \qquad \underline{\text{If}} \int |\psi_0(x)|^2 dx = 1$$

$$D_\alpha^* \psi_0^*(x) = N^* \psi_\alpha^*(x)$$

$$\underbrace{\int \left(D_\alpha \psi_0(x)\right)^* \left(D_\alpha \psi_0(x)\right) dx}_{\int \psi_0^*(x) \underbrace{D_\alpha^\dagger D_\alpha}_{\mathbb{1}} \psi_0(x)\, dx} = |N| \int |\psi_\alpha(x)|^2 dx$$

$$\int \psi_0^*(x) \underbrace{D_\alpha^\dagger D_\alpha}_{\mathbb{1}} \psi_0(x)\, dx = \int \psi_0^*(x)\psi_0(x)\, dx = 1$$

$$\Rightarrow N = 1 \quad \checkmark$$

$$\therefore \quad \psi_\alpha(x) = D_\alpha \psi_0(x) \qquad \langle \psi_\alpha | \psi_\alpha \rangle = 1$$

$$\langle \alpha | \beta \rangle = \int \psi_\alpha^*(x) \psi_\beta(x)\, dx$$

Note: $\langle \alpha | \beta \rangle \neq 0$

Note: $D_\alpha \psi_\beta(x) = ?$

$$\hat{a} D_\alpha \psi_\beta(x) = D_\alpha (\hat{a} + \alpha)\psi_\beta(x)$$
$$\hat{a}\psi_\beta(x) = \beta \psi_\beta(x)$$

$$\hat{a} D_\alpha \psi_\beta(x) = D_\alpha (\alpha + \beta)\psi_\beta(x)$$
$$= (\alpha + \beta) D_\alpha \psi_\beta(x)$$

$$\Rightarrow D_\alpha \psi_\beta(x) = \psi_{\alpha+\beta}(x) \quad \Rightarrow D_\alpha D_\beta = D_{(\alpha+\beta)}$$

Homework: $e^{\hat{A}} \hat{B} e^{-\hat{A}} = \sum_n \frac{1}{n!} [\hat{A}, \hat{B}]_{(n)}$

$\hookrightarrow$ nth sequential commutator

Homework: $e^{\hat{A}} \hat{B} e^{-\hat{A}} = \sum_n \frac{1}{n!} [A, B]_{(n)}$ ↪ nth sequential commutator

$[\hat{A}, \hat{B}]_{(0)} = \hat{B}$ $\qquad$ $[\hat{A}, \hat{B}]_{(2)} = [\hat{A}, [\hat{A}, \hat{B}]_-]_-$

$[\hat{A}, \hat{B}]_{(1)} = [\hat{A}, \hat{B}]_-$ $\qquad$ $[\hat{A}, \hat{B}]_{(n)} = [\hat{A}, [\hat{A}, \hat{B}]_{(n-1)}]_-$

Applying this

$$D_\alpha^\dagger \hat{a} D_\alpha = e^{\hat{A}} \hat{a} e^{-\hat{A}} \qquad \hat{A} = -\alpha a^\dagger + \alpha^* \hat{a}$$

$$[\hat{A}, \hat{a}]_{(0)} = \hat{a} \quad , \quad [\hat{A}, \hat{a}]_{(1)} = [-\alpha a^\dagger + \alpha^* \hat{a}, a]$$

$$= \alpha$$

$$[\hat{A}, \hat{a}]_{(2)} = [\hat{A}, \alpha]_- = 0$$

$$\therefore D_\alpha^\dagger \hat{a} D_\alpha = \hat{a} + \alpha + 0 + 0 + \cdots$$

Q.E.D.