Class 20 Coherent States and the Classical Limit

Thursday, November 8, 2018 8:00 AM

$$\hat{a}\,\psi_\alpha(x) = \alpha\,\psi_\alpha(x)$$

w/ $\psi_\alpha(x) = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{n!} (a^+)^n \psi_0(x)$ $\qquad \psi_0(x) = A\, e^{-\xi^2/2}$; $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$

$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \psi_n(x) \; ; \qquad C_n = \frac{e^{-|\alpha|^2/2}\, \alpha^n}{\sqrt{n!}}$$

$\hat{H}\psi_n(x) = E_n \psi_n(x)$

$E_n = \hbar\omega(n + 1/2)$

Claim:

$\langle \hat{x} \rangle_{\alpha,t} = X(t)$ $\qquad$ where for the classical H.O.

$\langle \hat{p} \rangle_{\alpha,t} = P(t)$

$$\left\{ \begin{aligned} X(t) &= X(0)\cos\omega t \\ \frac{P(t)}{m} &= X(0)\,\omega\, \sin\omega t \end{aligned} \right\}$$

we had found that for the H.O.

$$\left. \begin{aligned} \langle \hat{x} \rangle_t &= \langle \hat{x} \rangle_0 \cos\omega t + \frac{\langle \hat{p} \rangle_0}{m\omega} \sin\omega t \\ \langle \hat{p} \rangle_t &= -m\omega \langle \hat{x} \rangle_0 \sin\omega t + \langle \hat{p} \rangle_0 \cos\omega t \end{aligned} \right\}$$

This satisfies the Ehrenfest theorem:

$$\frac{d\langle \hat{x} \rangle_t}{dt} = \frac{\langle \hat{p} \rangle_t}{m} \iff \frac{dX(t)}{dt} = V(t) = \frac{P(t)}{m}$$

while this is true for any state of the H.O. we want to show that for "Coherent States"

$\langle \hat{X} \rangle_t = X(t)$ $\qquad$ "In the Classical Limit"

$\langle \hat{P} \rangle_t = P(t)$

$$\langle \hat{x} \rangle_{\alpha, t=0} = \int \psi_\alpha^*(x,t)\, \hat{x}\, \psi_\alpha(x,t)\, dx \Big|_{t=0} = \sqrt{\frac{\hbar}{2m\omega}} \int \psi_\alpha^*(x,0) \left(\hat{a} + \hat{a}^+\right) \psi_\alpha(x,0)\, dx$$

$$\langle \hat{x} \rangle_{\alpha,0} = \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha + \alpha^*\right)$$

Likewise,

$$\langle \hat{p} \rangle_{\alpha,0} = -i\sqrt{\frac{\hbar m\omega}{2}} \int \psi_\alpha^* (\hat{a} - \hat{a}^\dagger)\psi_\alpha \, dx$$

$$= -i\sqrt{\frac{\hbar m\omega}{2}} (\alpha - \alpha^*)$$

$$\langle \hat{x} \rangle_{\alpha,t} = \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*) \cos\omega t - i\sqrt{\frac{\hbar m\omega}{2}} \frac{1}{m\omega} (\alpha - \alpha^*) \sin\omega t$$

$$\langle \hat{p} \rangle_{\alpha,t} = -m\omega \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*) \sin\omega t - i\sqrt{\frac{\hbar m\omega}{2}} (\alpha - \alpha^*) \cos\omega t$$

$$\langle \hat{x} \rangle_{\alpha,t} = \sqrt{\frac{\hbar}{2m\omega}} \left[(\alpha + \alpha^*) \cos\omega t - i(\alpha - \alpha^*) \sin\omega t \right]$$

$$\langle \hat{p} \rangle_{\alpha,t} = -\sqrt{\frac{\hbar m\omega}{2}} \left[(\alpha + \alpha^*) \sin\omega t + i(\alpha - \alpha^*) \cos\omega t \right]$$

$$\alpha = |\alpha| e^{i\varphi} \; ; \; \alpha^* = |\alpha| e^{-i\varphi}$$

$$\alpha + \alpha^* = 2|\alpha| \cos\varphi \qquad \alpha - \alpha^* = i\,2|\alpha| \sin\varphi$$

$$\langle \hat{x} \rangle_{\alpha,t} = \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \left[\cos\varphi \cos\omega t + \sin\varphi \sin\omega t \right]$$

$$\langle \hat{p} \rangle_{\alpha,t} = -\sqrt{2\hbar m\omega} |\alpha| \left[\cos\varphi \sin\omega t - \sin\varphi \cos\omega t \right]$$

$$\langle \hat{x} \rangle_{\alpha,t} = \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos(\omega t - \varphi) \qquad \langle \hat{x} \rangle_{\alpha,0} = \sqrt{\frac{2\hbar}{m\omega}} |\alpha| \cos\varphi$$

$$\langle \hat{p} \rangle_{\alpha,t} = -\sqrt{2\hbar m\omega} |\alpha| \sin(\omega t - \varphi) \qquad \langle \hat{p} \rangle_{\alpha,0} = +\sqrt{2\hbar m\omega} |\alpha| \sin\varphi$$

Note: the phase of α is determined by the initial conditions!

For the Classical H. O. $E = T + V$

$$= \frac{p^2(0)}{2m} + \frac{1}{2} m\omega^2 x^2(0)$$

Using Ehrenfest thm:

$$\langle \hat{H} \rangle_{class} = \frac{\langle p \rangle_{\alpha,0}^2}{2m} + \frac{1}{2} m\omega^2 \langle x \rangle_{\alpha,0}^2$$

"Classical correspondence"

$$\langle \hat{H} \rangle_{\alpha,0} = \frac{2\hbar m\omega}{2m} |\alpha|^2 \sin^2\varphi + \frac{1}{2} m\omega^2 \frac{2\hbar}{m\omega} |\alpha|^2 \cos^2\varphi$$

$$\langle \hat{H} \rangle_{class} = \frac{\cancel{2}\hbar m\omega}{\cancel{2}m} |\alpha|^2 \sin^2\varphi + \frac{1}{\cancel{2}} m\omega^2 \frac{\cancel{2}\hbar}{m\omega} |\alpha|^2 \cos^2\varphi$$

$$= \hbar\omega |\alpha|^2 \left(S^2\varphi + \cos^2\varphi \right)$$

$$\langle \hat{H} \rangle_{class} = \hbar\omega |\alpha|^2$$

Compare w/ the Quantum result:

$$\langle \hat{H} \rangle = \sum_n |C_n|^2 E_n = \hbar\omega \sum_n |C_n|^2 \left(n + 1/2\right)$$

$$= \hbar\omega \sum_n n|C_n|^2 + \frac{\hbar\omega}{2}$$

$$= \hbar\omega \langle n \rangle + \frac{\hbar\omega}{2}$$

$$\langle \hat{H} \rangle = \hbar\omega |\alpha|^2 + \frac{\hbar\omega}{2} \qquad \text{off by } \frac{\hbar\omega}{2}$$

$$\langle \hat{H} \rangle \simeq E_{class} \quad \checkmark$$

$\therefore$ The Coherent States w/ $|\alpha|$ large properly describe the classical regime.

We expect that for the Coherent States $\Delta X \Delta P = \hbar/2$, the product is as small as it is allowed to be.

$$\langle \hat{X} \rangle_{\alpha,0} = \sqrt{\frac{\hbar}{2m\omega}} \left(\alpha + \alpha^*\right)$$

$$\langle \hat{X}^2 \rangle_{\alpha,0} = \frac{\hbar}{2m\omega} \langle (\hat{a} + \hat{a}^+)^2 \rangle_{\alpha,0} = \frac{\hbar}{2m\omega} \langle \hat{a}^2 + \hat{a}^{+2} + \hat{a}\hat{a}^+ + \hat{a}^+\hat{a} \rangle_{\alpha,0}$$

$$= \frac{\hbar}{2m\omega} \langle \hat{a}^2 + \hat{a}^{+2} + 2\hat{a}^+\hat{a} + 1 \rangle_{\alpha,0}$$

$$= \frac{\hbar}{2m\omega} \left[\alpha^2 + \alpha^{*2} + 2\alpha^*\alpha + 1 \right]$$

$$= \frac{\hbar}{2m\omega} \left[(\alpha + \alpha^*)^2 + 1 \right]$$

$$\langle \hat{X}^2 \rangle_{\alpha,0} - \langle \hat{X} \rangle_{\alpha,0}^2 = \frac{\hbar}{2m\omega}$$

$$\langle \hat{P} \rangle_{\alpha,0} = -i\sqrt{\frac{\hbar m\omega}{2}} \left(\alpha - \alpha^*\right) \; ; \; \langle \hat{P}^2 \rangle_{\alpha,0} = -\frac{\hbar m\omega}{2} \left[(\alpha - \alpha^*)^2 - 1 \right]$$

$$\langle \hat{p}^2 \rangle_{\alpha,0} - \langle \hat{p} \rangle^2_{\alpha,0} = \frac{\hbar m \omega}{2}$$

$$\Delta X \, \Delta P = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{\hbar m \omega}{2}} = \hbar/2 \quad \checkmark$$

We expect $\psi_\alpha(x)$ is a gaussian (Homework)