Class #21 Dirac Formalism

Friday, November 9, 2018 8:29 AM

$$\hat{H}\psi_n(x) = E_n\psi_n(x)$$

for Any 2 states $\psi_1(x,0) = \sum_n c_n \psi_n(x)$

$$\left.\begin{array}{l}\psi_1(x,0) = \sum_n c_n \psi_n(x) \\ \psi_2(x,0) = \sum_n d_n \psi_n(x)\end{array}\right\} \text{Energy Eigenbasis}$$

$$\langle\psi_1(x)|\psi_2(x)\rangle = \sum_n c_n^* d_n$$

$$= \int \psi_1^*(x)\,\psi_2(x)\,dx$$

$\hat{Q}\phi_m = q_m\phi_m$: the ϕ's will also form a complete basis.

$$\Rightarrow \quad \psi_1(x) = \sum_n f_n^{(1)}\phi_n \qquad \left.\psi_2(x) = \sum_n f_n^{(2)}\phi_n\right\} \text{Eigenbasis of } \hat{Q}$$

$$\langle\psi_1(x)|\psi_2(x)\rangle = \sum_n f_n^{(1)*} f_n^{(2)}$$

$$= \sum_n c_n^* d_n$$

The Inner product is independent of basis!

Furthermore, $\psi_1(x)$ & $\psi_2(x)$ are characterized by either c_n & d_n's or $f_n^{(1)}, f_n^{(2)}$'s expansion coefficients.

Yet another example,

$$\left.\begin{array}{l}\psi_1(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_1(k)\,e^{ikx}\,dk \\ \psi_2(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_2(k)\,e^{ikx}\,dk\end{array}\right\} \text{Eigenbasis of the Momentum operator}$$

$$\langle\psi_1(x)|\psi_2\rangle = \int_{-\infty}^{\infty}\psi_1^*(x)\,\psi_2(x)\,dx = \frac{1}{2\pi}\iiint \phi_1^*(k)\,\phi_2(k')\,e^{-ikx}e^{ik'x}\,dk\,dk'\,dx$$

$$= \frac{1}{2\pi}\int dk\,dk'\,\phi_1^*(k)\,\phi_2(k')\int e^{i(k'-k)x}\,dx$$

$$\int \phi_1^*(k)\,\phi_2(\ldots)\,dk$$

$$2\pi \int \qquad \int \qquad dx$$

$$= \int \phi_1^*(k)\, \phi_2(k)\, dk$$

$$= \langle \phi_1(k) | \phi_2(k) \rangle$$

In Linear Algebra a vector $\vec{V}$ has components (V_x, V_y, V_z)

$$\vec{V} = \sum_i \hat{e}_i V_i = \hat{i} V_x + \hat{j} V_y + \hat{k} V_z$$

$$= \sum_i \hat{e}_i' V_i'$$

$\therefore$ Dirac found it convenient to invent a formalism where quantum states are defined indep. of basis.

To each quantum state $\longleftrightarrow$ $|\psi\rangle$ "Ket Vector"

$\langle\psi|$ "bra Vector"

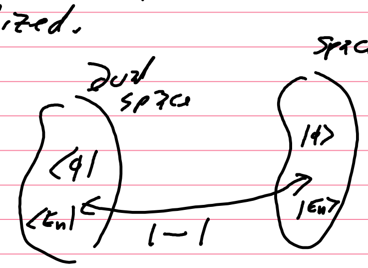
$$\hat{H}|E_n\rangle = E_n|E_n\rangle$$

$|E_n\rangle$ "Form a basis" this basis can always be orthogonalized.

$$\langle E_n|E_m\rangle = \delta_{mn}$$

$(\hat{e}_i)$

$$|\psi\rangle = \sum c_n |E_n\rangle$$



$$\langle E_m|E_n\rangle = \delta_{mn}$$

$$c_n = \langle E_n|\psi\rangle$$

$|c_n|^2$ = prob of obtaining E_n

$$\langle \hat{H} \rangle = \langle\psi|\hat{H}|\psi\rangle = \sum_{m,n} c_m^* c_n \langle E_m|\hat{H}|E_n\rangle$$

$$= \sum_n |c_n|^2 E_n$$

For any Operator $\hat{Q}$:

$$\hat{Q}|\psi\rangle = |\psi'\rangle$$

$$\hat{Q} \sum_n c_n |E_n\rangle = \sum_m c_m' |E_m\rangle$$

$$\sum c_n \hat{Q}|E_n\rangle = \sum_m c_m' |E_m\rangle$$

$$\sum_n c_n \langle E_l|\hat{Q}|E_n\rangle = \sum_m c_m' \langle E_l|E_m\rangle$$

$$\sum_n \underbrace{\langle E_l|\hat{Q}|E_n\rangle}\, c_n = c_l'$$

$$\sum_n \hat{Q}_{ln}\, c_n = c_l'$$

$$\begin{array}{c|c} & n = 1 \quad 2 \quad \cdots \\ \begin{matrix} l \\ 1 \\ 2 \\ \vdots \end{matrix} & \begin{pmatrix} Q_{11} & Q_{12} & \cdots \\ Q_{21} & Q_{22} & \cdots \\ \vdots & \vdots & Q_{33} \\ & & \ddots \end{pmatrix} \end{array} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} c_1' \\ c_2' \\ \vdots \end{pmatrix}$$

If $\hat{Q}$ is Hermitian

$$\Rightarrow \langle E_l|\hat{Q}|E_n\rangle^* = \langle E_n|\hat{Q}^\dagger|E_l\rangle$$

$$= \langle E_n|\hat{Q}|E_l\rangle$$

$$\hat{Q}_{ln}^* = \hat{Q}_{nl} \quad \Rightarrow \quad \underset{\sim}{\hat{Q}}^{*T} = \underset{\sim}{\hat{Q}}$$

Any operator can be written as the outer product of two ket vectors: of the form $|\varphi\rangle\langle\psi|$

In linear, $\hat{e}_i \otimes \hat{e}_j$

e.g. $\hat{Q}|\psi\rangle = |\psi'\rangle \Rightarrow \hat{Q} = |\psi'\rangle\langle\psi|$

$\hat{\mathbb{P}} \equiv |\psi\rangle\langle\psi|$ "Projection operator"

If this operator acts of any ket vector $|\varphi\rangle$:

$$\hat{\mathbb{P}}|\varphi\rangle = |\psi\rangle\langle\psi|\varphi\rangle = c|\psi\rangle$$

$$\hat{\mathbb{P}}^2 = \left(|\psi\rangle\langle\psi|\right)\left(|\psi\rangle\langle\psi|\right) = \mathbb{P}$$

Projection operators for basis vectors:

$$\mathbb{P}_n = |E_n\rangle\langle E_n| \qquad \mathbb{P}_n^2 = \mathbb{P}_n$$

$$\mathbb{P}_n|\psi\rangle = |E_n\rangle\langle E_n|\psi\rangle$$

$$= c_n|E_n\rangle$$

$$\sum_n \mathbb{P}_n|\psi\rangle = \sum_n c_n|E_n\rangle = |\psi\rangle$$

$$\sum_n \mathbb{P}_n = \mathbb{1} \qquad \text{"Completeness"}$$

$$\hookrightarrow \sum_n \psi_n(x)\,\psi_n(x') = \delta(x-x')$$

$$\mathbb{P}_n \mathbb{P}_m = \mathbb{P}_n \delta_{mn}$$

$|\psi\rangle$ & $|\varphi\rangle$ the inner product $\langle\varphi|\psi\rangle = \langle\varphi|\mathbb{1}|\psi\rangle$

$$\langle\varphi|\psi\rangle = \sum_n \langle\varphi|\mathbb{P}_n|\psi\rangle = \sum_n \underbrace{\langle\varphi|E_n\rangle}_{d_n^*}\underbrace{\langle E_n|\psi\rangle}_{c_n}$$

$$= \sum_n d_n^* c_n$$

$$\mathbb{P}_n^{(a)} = |a_n\rangle\langle a_n|$$

Basis Labeled by Continuous Index:

$$\hat{X}|x\rangle = x|x\rangle \qquad \langle x|x'\rangle = 0 \quad x \neq x'$$

$$\mathbb{P}_x = |x\rangle\langle x| \qquad \mathbb{P}_x^2 = \mathbb{P}_x$$

$$\int \mathbb{P}_x\, dx = \int dx\, |x\rangle\langle x| = \mathbb{1}$$

$$|\psi\rangle = \int dx\, c(x)\, |x\rangle$$

$$\langle x'|\psi\rangle = \int dx\, c(x)\, \langle x'|x\rangle \Rightarrow \langle x'|x\rangle = \delta(x-x')$$

$$c(x') = \langle x'|\psi\rangle$$

c.f. $\quad c_n = \langle E_n|\psi\rangle$

$$\int dx\, \langle y'|x\rangle\langle x|y\rangle = \langle y'|y\rangle$$

$$\Rightarrow \int dx\, \delta(x-y')\,\delta(y-x) = \delta(y'-y) \quad \checkmark$$

$$|\varphi\rangle = \int dy \; f(y)\,|y\rangle \qquad \langle\varphi| = \int dy \; \langle y|\, f^*(y)$$

$$\langle\varphi|\psi\rangle = \int dx\,dy \; f^*(y)\; c(x) \; \underbrace{\langle y|x\rangle}_{\delta(x-y)}$$

"Coordinate basis"

$$\left.\begin{aligned} &= \int dx \; f^*(x)\; c(x) \\ &= \int dx \; \varphi^*(x)\; \psi(x) \end{aligned}\right\} \Rightarrow \begin{aligned} f(x) &= \varphi(x) \\ c(x) &= \psi(x) \end{aligned}$$

<u>Momentum Basis</u>

$$\hat{P}|p\rangle = p|p\rangle \qquad |\psi\rangle = \int dp \; \mathbb{P}_p \,|\psi\rangle$$

$$\mathbb{P}_p = |p\rangle\langle p| \qquad = \int dp \; |p\rangle\langle p|\psi\rangle$$

$$\int \mathbb{P}_p \, dp = \mathbb{1} \qquad = \int dp \; \varphi(p)\; |p\rangle$$

$$\langle x|\psi\rangle = \int dp \; \varphi(p)\; \langle x|p\rangle$$

$$\psi(x) = \int dp \; \varphi(p)\; \langle x|p\rangle \longleftrightarrow \psi(x) = \frac{1}{\sqrt{2\pi}} \int dk \; \varphi(k)\, e^{ikx}$$

$$\langle x|p\rangle \sim e^{ikx} \qquad p = \hbar k$$