Two State Systems Continued

(Send chat if you do not hear me)

The most quantum state is characterized by a ket vector $|\psi\rangle$:

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle$$

$|1\rangle, |2\rangle$

$\langle 1|2\rangle = 0$

$\langle 1|1\rangle = 1$

$\langle 2|2\rangle = 1$

Exchange Operator:

$$\hat{E} = |1\rangle\langle 2| + |2\rangle\langle 1| \quad *$$

$$\hat{E}|1\rangle = |2\rangle \quad ; \quad \hat{E}|2\rangle = |1\rangle$$

$$\hat{E}^2 = \mathbb{1} \qquad \hat{E}^2|1\rangle = |1\rangle \quad , \quad \hat{E}^2|2\rangle = |2\rangle$$

$$\underset{\sim}{\hat{E}}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; * \qquad \underset{\sim}{\hat{E}}^2 = \begin{pmatrix} \langle 1|\hat{E}^2|1\rangle & \langle 1|\hat{E}|2\rangle \\ \langle 2|\hat{E}^2|1\rangle & \langle 2|\hat{E}|2\rangle \end{pmatrix}$$

$$\hat{E}^2 = \left(|1\rangle\langle 2| + |2\rangle\langle 1|\right)\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right)$$

$$= |1\rangle\cancelto{0}{\langle 2|1\rangle}\langle 2| + |1\rangle\langle 2|2\rangle\langle 1| + |2\rangle\langle 1|1\rangle\langle 2| + |2\rangle\cancelto{0}{\langle 1|2\rangle}\langle 1|$$

$$\hat{E}^2 = \underbrace{|1\rangle\langle 1| + |2\rangle\langle 2|}_{} = \sum_{i=1}^{2} \mathbb{P}_i = \mathbb{1}$$

$$= \sum_{i=1}^{2} |i\rangle\langle i| = \mathbb{1}$$

The Eigenvalues of $\hat{E}$ can have the form:

$$e^{i\varphi} \qquad \text{since } |e^{i\varphi}|^2 = 1$$

However, $\hat{E}^\dagger = \hat{E}$

$$\hat{E}^\dagger = \left(|1\rangle\langle 2|\right)^\dagger + \left(|2\rangle\langle 1|\right)^\dagger$$

$$= |2\rangle\langle 1| + |1\rangle\langle 2| = \hat{E} \Rightarrow$$

Eigenvalues must be real

$\Rightarrow \pm 1$

What are the eigenkets of $\hat{E}$?

$$\boxed{\hat{E}|\pm\rangle = \pm|\pm\rangle}$$

$$|\pm\rangle = c_1^{\pm}|1\rangle + c_2^{\pm}|2\rangle$$

$$\hat{\mathbb{E}}\left(c_1^\pm|1\rangle + c_2^\pm|2\rangle\right) = \pm\left(c_1^\pm|1\rangle + c_2^\pm|2\rangle\right)$$

$$c_1^\pm|2\rangle + c_2^\pm|1\rangle = \pm\left(c_1^\pm|1\rangle + c_2^\pm|2\rangle\right)$$

$\langle 2|\circledast$

$$c_1^\pm = \pm\left(0 + c_2^\pm\right)$$

$$c_1^\pm = \pm c_2^\pm \qquad c_1^+ = c_2^+$$

$$c_1^- = -c_2^-$$

$$|+\rangle = c_1^+\left(|1\rangle + |2\rangle\right) \qquad c_1^+ = \qquad \langle +|+\rangle = 1$$

$$|-\rangle = c_1^-\left(|1\rangle - |2\rangle\right) \qquad c_1^- = \qquad \langle -|-\rangle = 1$$

$$\langle +|+\rangle = |c_1^+|^2\left(\langle 2| + \langle 1|\right)\left(|1\rangle + |2\rangle\right)$$

$$= |c_1^+|^2\left(\langle 2|2\rangle + \langle 1|1\rangle\right) = 2|c_1^+|^2 = 1$$

$$\Rightarrow |c_1^+| = 1/\sqrt{2}$$

Likewise, $c_1^- = 1/\sqrt{2}$

The Eigenkets of $\hat{\mathbb{E}}$ are:

$$|\pm\rangle = \frac{|1\rangle \pm |2\rangle}{\sqrt{2}}$$

$\langle +|-\rangle = 0$

$$\begin{cases} |1\rangle = \dfrac{|+\rangle + |-\rangle}{\sqrt{2}} \\ |2\rangle = \dfrac{|+\rangle - |-\rangle}{\sqrt{2}} \end{cases}$$

$$\hat{\mathbb{E}}|\pm\rangle = \pm|\pm\rangle$$

"In its own basis"

$$\hat{\mathbb{E}} = \sum_i \lambda_i |i\rangle\langle i| = \underbrace{|+\rangle\langle +| - |-\rangle\langle -|}$$

$$\underset{\sim}{\hat{\mathbb{E}}} = \begin{pmatrix} \langle +|\hat{\mathbb{E}}|+\rangle & \langle +|\hat{\mathbb{E}}|-\rangle \\ \langle -|\hat{\mathbb{E}}|+\rangle & \langle +|\hat{\mathbb{E}}|+\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \checkmark (\pm)$$

Eigenvalues

Recall that any operator $\hat{Q}$

$$\underbrace{\hat{Q} = \sum_n q_n |q_n\rangle\langle q_n|} \; ; \quad \sum |q_n\rangle\langle q_n| = \mathbb{1}$$

$\hat{\mathbb{E}}$ in the $|1\rangle, |2\rangle$ basis:

$$\langle 1|\hat{\mathbb{E}}|\ldots \qquad \hat{\mathbb{E}} = \begin{pmatrix} \langle 1|\hat{\mathbb{E}}|1\rangle & \langle 1|\hat{\mathbb{E}}|2\rangle \\ \ldots & \ldots \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \ldots & \ldots \end{pmatrix} \checkmark$$

$\hat{E}$ in the $|1\rangle\,|2\rangle$ basis:

$$\langle i|\hat{E}|j\rangle \Rightarrow \underset{\sim}{\hat{E}} = \begin{pmatrix} \langle 1|\hat{E}|1\rangle & \langle 1|\hat{E}|2\rangle \\ \langle 2|\hat{E}|1\rangle & \langle 2|\hat{E}|2\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \checkmark$$

Lets define an operator T_+ such that $\hat{T}_+ \underline{|-\rangle} = \underline{|+\rangle}$

$$\hat{T}_+ \equiv |+\rangle\langle -| \quad ; \quad \hat{T}_- \equiv |-\rangle\langle +|$$

$$\hat{T}_+|+\rangle = |-\rangle ; \quad \hat{T}_+|+\rangle = 0 ; \qquad \langle +|-\rangle = 0$$

$$\hat{T}_-|+\rangle = |-\rangle \qquad \hat{T}_-|-\rangle = 0$$

$$[\hat{E}, \hat{T}_+] = \hat{E}\hat{T}_+ - \hat{T}_+\hat{E} = \big(|+\rangle\langle +| - |-\rangle\langle -|\big)\big(|+\rangle\langle -|\big)$$

$$- \big(|+\rangle\langle -|\big)\big(|+\rangle\langle +| - |-\rangle\langle -|\big)$$

$$= |+\rangle\langle -| + |+\rangle\langle -|$$

$$= 2|+\rangle\langle -|$$

$$\boxed{[\hat{E}, \hat{T}_+] = 2\hat{T}_+}$$

$$[\hat{E}, \hat{T}_-] = \big(|+\rangle\langle +| - |-\rangle\langle -|\big)\big(|-\rangle\langle +|\big)$$

$$- \big(|-\rangle\langle +|\big)\big(|+\rangle\langle +| - |-\rangle\langle -|\big)$$

$$= -|-\rangle\langle +| - |-\rangle\langle +| = -2\hat{T}_-$$

$$\boxed{[\hat{E}, \hat{T}_-] = -2\hat{T}_-}$$

$$[\hat{T}_+, \hat{T}_-] = \big(|+\rangle\langle -|\big)\big(|-\rangle\langle +|\big) - \big(|-\rangle\langle +|\big)\big(|+\rangle\langle -|\big)$$

$$= |+\rangle\langle +| - |-\rangle\langle -|$$

$$\boxed{[\hat{T}_+, \hat{T}_-] = \hat{E}}$$

$\hat{T}_+, \hat{T}_-, \hat{E}$ form a closed algebra!

Defn: $\hat{T}_1 \equiv \dfrac{\hat{T}_+ + \hat{T}_-}{2}$; $\hat{T}_2 \equiv \dfrac{\hat{T}_+ - \hat{T}_-}{2i}$ }

$\hat{T}_2 \equiv \hat{E}/2$ $\checkmark$ $\qquad \hat{E} \qquad -\hat{E}$

$$\hat{T}_3 \equiv \hat{\mathbb{E}}/2$$

$$[\hat{T}_1, \hat{T}_2] = \left[\frac{\hat{T}_+ + \hat{T}_-}{2}, \frac{\hat{T}_+ - \hat{T}_-}{2i}\right] = \frac{1}{4i}\left\{-\overbrace{[\hat{T}_+, \hat{T}_-]}^{\hat{\mathbb{E}}} + \overbrace{[\hat{T}_-, \hat{T}_+]}^{-\hat{\mathbb{E}}}\right\}$$

$$[\hat{T}_1, \hat{T}_2] = -\frac{1}{2i}\hat{\mathbb{E}} = i\hat{T}_3$$

$$[\hat{T}_1, \hat{T}_2] = i\hat{T}_3 \quad ; \quad [\hat{T}_3, \hat{T}_1] = i\hat{T}_2$$

$$[\hat{T}_2, \hat{T}_3] = i\hat{T}_1$$

In Summary, $\boxed{[\hat{T}_i, \hat{T}_j] = i\,\epsilon_{ijk}\,\hat{T}_k}$ * The operators $\hat{T}_1, \hat{T}_2$ & $\hat{T}_3$ form a close algebra

This is the same commutation relation of the angular momentum operators!

$$[\hat{L}_i, \hat{L}_j] = i\,\epsilon_{ijk}\,\hat{L}_k \; * \qquad \hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{P}}$$

$$[r_i, P_j] = i\hbar\,\delta_{ij}$$

In the $|\pm\rangle$ basis:

$$\underset{\sim}{T_3} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; * \; ; \; \hat{T}_1 = \frac{\hat{T}_+ + \hat{T}_-}{2} = \frac{|+\rangle\langle -| + |-\rangle\langle +|}{2} \; ; \; \hat{T}_2 = \frac{\hat{T}_+ - \hat{T}_-}{2i}$$

$$\underset{\sim}{T_1} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \underset{\sim}{T_2} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\underset{\sim}{T_i} = \frac{1}{2}\sigma_i \qquad \sigma_i : \text{Pauli Matrices}$$

$$[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\,\sigma_k \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\,\sigma_k$$

$$\{\sigma_i, \sigma_j\} = 2\,\delta_{ij}$$

In the T's in the $|1\rangle, |2\rangle$ basis:

$$\hat{T}_1 = \frac{\hat{T}_+ + \hat{T}_-}{2} = \frac{|+\rangle\langle -| + |-\rangle\langle +|}{2} \; *$$

$$|+\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}} \; ; \; |-\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$$ Eigenkets of $\hat{T}_3$

$$\boxed{|+\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}} \; ; \; \boxed{|-\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}} \Bigg\} \text{ of } \hat{T}_3$$

$$|+\rangle\langle -| = \left(\frac{|1\rangle + |2\rangle}{\sqrt{2}}\right)\left(\frac{\langle 1| - \langle 2|}{\sqrt{2}}\right)$$

$$= \frac{|1\rangle\langle 1| - |1\rangle\langle 2| + |2\rangle\langle 1| - |2\rangle\langle 2|}{2}$$

$$|-\rangle\langle +| = \left(\frac{|1\rangle - |2\rangle}{\sqrt{2}}\right)\left(\frac{\langle 1| + \langle 2|}{\sqrt{2}}\right)$$

$$= \frac{|1\rangle\langle 1| + |1\rangle\langle 2| - |2\rangle\langle 1| - |2\rangle\langle 2|}{2}$$

$$\boxed{\hat{T}_1 = |1\rangle\langle 1| - |2\rangle\langle 2|} \; *$$

$$\underset{\sim}{T_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; * \qquad \left.\begin{aligned} \hat{T}_1 |1\rangle &= |1\rangle \\ \hat{T}_1 |2\rangle &= -|2\rangle \end{aligned}\right\} \text{ Eigenkets of } \hat{T}_1$$

$$\boxed{|+\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}} \; ; \; |-\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$$

$$|1\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \; ; \; |2\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

In the $|\pm\rangle$ basis $|+\rangle$ is the column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|-\rangle$ is the column vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\hat{T}_3 |\pm\rangle = \pm |\pm\rangle$

$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$T_3 \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$

$|\pm\rangle$ in the $|1\rangle, |2\rangle$ basis:

$$|+\rangle = \begin{pmatrix} \langle 1|+\rangle \\ \langle 2|+\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In Q.M. $|\psi\rangle$, $e^{i\phi}|\psi\rangle$ both rep. the same state

$$\langle +|+\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \qquad \langle -|-\rangle = \begin{pmatrix} 0 & \pm 1 \end{pmatrix}\begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = 1$$

$$\langle +|-\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \; \end{pmatrix} = x$$

$\langle + | - \rangle = (1 \;\; 0) \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} = 0$ $\quad = 1$

$\boxed{\hat{T}_+ = \pm |+\rangle\langle -|}$ $\quad \underset{\sim}{T_+} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix}$ $\quad \hat{T}_+ |-\rangle = -|+\rangle$

$$\underset{\sim}{T_+} = \underbrace{\begin{pmatrix} \overset{0}{\langle + | + \rangle \langle - | + \rangle} & \overset{1}{\langle + | + \rangle \langle - | - \rangle} \\ 0 & 0 \end{pmatrix}} = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} =$$

$$\underset{\sim}{T_+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{matrix} 0 & -1 \\ 0 & 0 \end{matrix}$$

$$\hat{T}_+ = \begin{cases} |+\rangle\langle -| \rightsquigarrow & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad |\pm\rangle \text{ basis} \\ |1\rangle\langle 1| - |2\rangle\langle 2| & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad |1\rangle, |2\rangle \text{ basis} \end{cases}$$

The eigen kets of $\hat{T}_1$ is $\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}$

$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ transforms from the $|\pm\rangle$ basis to $|1\rangle, |2\rangle$, basis.

$$\hat{T}_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow S \underset{\sim}{\hat{T}_3} S^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{2}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S \hat{T}_3 S^{-1} \longrightarrow \underbrace{\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\hat{T}_1}$$

The Hamiltonian in a two state system:

$$\hat{H} = \sum_{i,j} H_{ij} |i\rangle\langle j| = H_{11} |1\rangle\langle 1| + H_{22} |2\rangle\langle 2|$$
$$H_{12} |2\rangle\langle 1| + H_{21} |1\rangle\langle 2|$$

$$\underbrace{\underset{\sim}{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{22} & H_{22} \end{pmatrix}} \qquad \underset{\sim}{\hat{H}}^{+} = \underset{\sim}{\hat{H}} \Rightarrow \underbrace{\begin{matrix} H_{11} = H_{11}^{*} & H_{12}^{*} = H_{21} \\ H_{22} = H_{22}^{*} & \end{matrix}}_{**}$$

$$H_{ij} = \langle i | \hat{H} | j \rangle$$