Class 24 Hamiltonians of Two State SYstems

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We have seen that in some sense we can characterized two state systems by three operators, $\hat{T}_1, \hat{T}_2, \hat{T}_3$

$$\ni \quad [T_i, T_j] = i\epsilon_{ijk}T_k$$

Since they do not commute we cannot diagonalize all three simultaneously

$$[T_iT_i, T_j] = [\hat{T}^2, T_j] = 0 \qquad T_i^2 = 1 \quad \forall i$$

$$\hat{T}^2 = T_1^2 + T_2^2 + T_3^2 \quad \& \text{ say } \hat{T}_3$$

$$\sigma_i^\dagger = \sigma_i$$

Note: $T_i = \frac{1}{2}\sigma_i \qquad \sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k \;;\; tr(\sigma_i) = 0$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ;\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ;\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad tr(\sigma_i\sigma_j) = 2\delta_{ij}$$

For a 2-state system the Hamiltonian has of course two eigenstates with 2-corresponding eigenvalues. However in a general basis $\hat{H}$ can be written as:

$$\hat{H} = \sum_{i} H_{ij}|i\rangle\langle j| = H_{11}|1\rangle\langle 1| + H_{22}|2\rangle\langle 2| + H_{12}|1\rangle\langle 2| + H_{21}|2\rangle\langle 1|$$

In the $|1\rangle, |2\rangle$ basis:

$$\hat{H}^\dagger = \hat{H} \Rightarrow H_{11} \,\&\, H_{22} \text{ Real}, \quad H_{12}^* = H_{21}$$

$$\underset{\sim}{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$$

In its own basis; $\underset{\sim}{H} = \begin{pmatrix} E_+ & 0 \\ 0 & E_- \end{pmatrix}$

$$\hat{H} = E_+|+\rangle\langle +| + E_-|-\rangle\langle -|$$

$$\cancel{\ast}\; \underbrace{\hat{H}|\pm\rangle = E_\pm|\pm\rangle} \;; \qquad |\pm\rangle = C_1^\pm|1\rangle + C_2^\pm|2\rangle$$

$$\Rightarrow \underset{\sim}{H}\begin{pmatrix} C_1^\pm \\ C_2^\pm \end{pmatrix} = E_\pm\begin{pmatrix} C_1^\pm \\ C_2^\pm \end{pmatrix} \qquad \underset{\sim}{H}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \lambda\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

$$\begin{pmatrix} H_{11}-\lambda & H_{12} \\ H_{21} & H_{22}-\lambda \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

To avoid the trivial solution we require,

$$\begin{vmatrix} H_{11}-\lambda & H_{12} \\ H_{21} & H_{22}-\lambda \end{vmatrix} = 0$$

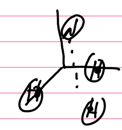
$$(H_{11}-\lambda)(H_{22}-\lambda) - H_{12}H_{21} = 0$$

$$\lambda^2 - (H_{11} + H_{22})\lambda + H_{11}H_{22} - |H_{12}|^2 = 0$$

$\Rightarrow$ Has two solutions $\quad \lambda = \dfrac{H_{11}+H_{22}}{2} \pm \dfrac{1}{2}\left[(H_{11}+H_{22})^2 - 4H_{11}H_{22} + 4|H_{12}|^2\right]^{1/2}$

$$E_\pm = \frac{H_{11}+H_{22}}{2} \pm \frac{1}{2}\left[(H_{11}-H_{22})^2 + 4|H_{12}|^2\right]^{1/2}$$

$$E_0 = \frac{H_{11}+H_{22}}{2} \qquad A = \frac{1}{2}\left[(H_{11}-H_{22})^2 + 4|H_{12}|^2\right]^{1/2}$$



$$H_{12} = \langle 1|H|2\rangle = \int \psi_1^*(\vec{x})\,\hat{H}\,\psi_2(\vec{x})\,d^3x$$

$$E_\pm = E_0 \pm A \qquad \underset{\sim}{H} = \begin{pmatrix} E_0 + A & 0 \\ 0 & E_0 - A \end{pmatrix}$$

$$\underset{\sim}{H} = E_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = E_0 \mathbb{1} + A\sigma_3$$

Note: $\quad [\underset{\sim}{H}, \sigma_3] = 0 \qquad \boxed{\underset{\sim}{H} = E_0\mathbb{1} + A\sigma_3}$

In general any 2×2 matrix can be written in the form: $\hat{Q} = Q_0\mathbb{1} + \vec{Q}\cdot\vec{\sigma} \qquad \{\underset{\sigma_0}{\mathbb{1}}, \sigma_1, \sigma_2, \sigma_3\}$

In Particular, $\quad \hat{H} = h_0\mathbb{1} + \vec{h}\cdot\vec{\sigma}$

$$\hat{H}^\dagger = \hat{H} \Rightarrow h_0^* = h_0$$

$$h_0^*\mathbb{1} \quad h_i^*\sigma_i^\dagger = h_0\mathbb{1} + h_i\sigma_i$$

$$h_i^*\,tr(\sigma_j\cdot\sigma_i^\dagger) = h_i\,tr(\sigma_j\sigma_i) \qquad \sigma_1^* = \sigma_1$$

$$h_i^*\,tr(\sigma_j\cdot\sigma_i^\dagger) = 2h_j \qquad \sigma_2^* = -\sigma_2$$

$$\Rightarrow h_i^* = h_i \qquad \sigma_3^* = \sigma_3$$

$\Rightarrow \underset{\sim}{H}$ is characterized by 4 parameters (h_0, h_1, h_2, h_3)

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} h_0+h_3 & h_1 - ih_2 \\ h_1 + ih_2 & h_0 - h_3 \end{pmatrix} \qquad \begin{matrix} H_{11} = h_0 + h_3 & H_{12} = h_1 - ih_2 \\ H_{22} = h_0 - h_3 & \end{matrix}$$

$$\underset{\sim}{H} = h_0\mathbb{1} + h_j\sigma_j \qquad \left.\begin{matrix} h_0 = \frac{1}{2}tr\,\underset{\sim}{H} = \dfrac{H_{11}+H_{22}}{2} \\ h_i = \frac{1}{2}tr(\sigma_i\underset{\sim}{H}) \end{matrix}\right\}$$

Spse there is another observable in this 2 state system. Lets call the corresponding operator $\hat{W}$ and spse $[\hat{H}, \hat{W}] \neq 0$
In the basis that $\hat{H}$ is diagonalized $\hat{W}$ will not be diagonal.

$$\hat{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = \overline{W_0}\,\mathbb{1} + \overset{\frown}{\vec{W}\cdot\vec{\sigma}}$$

Note: $\hat{H} = E_0\mathbb{1} + A\sigma_3$
$[\hat{H}, \hat{W}] \neq 0$

$$\hat{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} = W_0 \mathbb{1} + \vec{W}\cdot\vec{\sigma} \qquad [\hat{H},\hat{W}] \neq 0$$

$$\vec{W}\cdot\vec{\sigma} \equiv (\hat{n}\cdot\vec{\sigma})W \; ; \qquad \hat{n} = \frac{\vec{W}}{|\vec{W}|}$$

We can always write $\hat{n}$ as:
$\hat{n} = (S\theta\, C\varphi, S\theta\, S\varphi, C\theta)$
$\hat{n}^2 = 1 \quad \theta$ & φ are just 2-parameters

The eigenvectors of $\hat{W}$ are the same as the eigenvectors of $\hat{n}\cdot\vec{\sigma}$

i.e. if $\quad (\hat{n}\cdot\vec{\sigma})|\xi\rangle = \xi|\xi\rangle$

then $\quad \hat{W}|\xi\rangle = (W_0 + \xi W)|\xi\rangle$

Note, $\quad (\hat{n}\cdot\vec{\sigma})^2 = \hat{n}_i \hat{n}_j \sigma_i \sigma_j = \hat{n}_i \hat{n}_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k)$

$= \mathbb{1}$ ✓

$\Rightarrow$ The eigenvalues of $\hat{n}\cdot\vec{\sigma}$ are ± 1

$$\hat{n}\cdot\vec{\sigma}|\pm\rangle = \pm|\pm\rangle \qquad \hat{n}\cdot\vec{\sigma} = \begin{cases} \sigma_1 & \theta = \pi/2, \varphi = 0 \\ \sigma_2 & \theta = \pi/2, \varphi = \pi/2 \\ \sigma_3 & \theta = 0 \end{cases}$$

$$\hat{n}\cdot\vec{\sigma} = \begin{pmatrix} \hat{n}_3 & \hat{n}_1 - i\hat{n}_2 \\ \hat{n}_1 + i\hat{n}_2 & -\hat{n}_3 \end{pmatrix} = \begin{pmatrix} C\theta & S\theta e^{-i\varphi} \\ S\theta e^{i\varphi} & -C\theta \end{pmatrix}$$

$$\det\begin{vmatrix} C\theta - \lambda & S\theta e^{-i\varphi} \\ S\theta e^{i\varphi} & \underline{-C\theta - \lambda} \end{vmatrix} = 0 \Rightarrow -(\lambda^2 - C^2\theta) - S^2\theta = 0$$

$$\lambda^2 = S^2\theta + C^2\theta = 1$$

$$\lambda = \pm 1 \checkmark$$

$$\begin{pmatrix} C\theta & S\theta e^{-i\varphi} \\ S\theta e^{i\varphi} & -C\theta \end{pmatrix}\begin{pmatrix} C_1^\pm \\ C_2^\pm \end{pmatrix} = \pm\begin{pmatrix} C_1^\pm \\ C_2^\pm \end{pmatrix}$$

$$\begin{pmatrix} C_1^+ \\ C_2^+ \end{pmatrix} = \begin{pmatrix} C\theta/2 \\ S\theta/2\, e^{i\varphi} \end{pmatrix} \qquad \begin{pmatrix} C_1^- \\ C_2^- \end{pmatrix} = \begin{pmatrix} S\theta/2\, e^{-i\varphi} \\ -C\theta/2 \end{pmatrix}$$

$$|+\rangle = \begin{pmatrix} C\theta/2 \\ S\theta/2\, e^{i\varphi} \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} S\theta/2\, e^{-i\varphi} \\ -C\theta/2 \end{pmatrix}$$

$$\langle +|-\rangle = (C\theta/2 \quad S\theta/2\, e^{-i\varphi})\begin{pmatrix} S\theta/2\, e^{-i\varphi} \\ -C\theta/2 \end{pmatrix}$$

$$= 0$$

$$\langle +|+\rangle = 1 \qquad \langle -|-\rangle = 1$$

Note: $\hat{n} = (0,0,1) \Rightarrow \theta = 0 \Rightarrow |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad |-\rangle = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\hat{n} = (1,0,0) \Rightarrow \theta = \pi/2, \varphi = 0 \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\hat{n} = (0,1,0) \Rightarrow \theta = \pi/2, \varphi = \pi/2 \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad |-\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\langle +|-\rangle = \frac{1}{2}(1 \;\; -i)\begin{pmatrix} i \\ 1 \end{pmatrix} = 0$$

σ_1: $\quad |\pm, x\rangle = \frac{1}{\sqrt{2}}\left(|+,z\rangle \pm |-,z\rangle\right)$

$\langle +|-\rangle = \frac{1}{2}(1\ -i)\binom{1}{i} = 0$

σ_1: $|\pm, x\rangle = \frac{1}{\sqrt{2}}\left(|+,z\rangle \pm |-,z\rangle\right)$

σ_2: $|\pm, y\rangle = \frac{1}{\sqrt{2}}\left(|+,z\rangle \pm i\,|-,z\rangle\right)$

σ_3: $|+,z\rangle, \ |-,z\rangle$

The Eigenkets of $\hat{W}$:

$$|+,W\rangle = \cos\theta/2\ |+,z\rangle + \sin\theta/2\ e^{i\varphi}|-,z\rangle \Rightarrow \cos\theta/2\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta/2\ e^{i\varphi}\begin{pmatrix}0\\1\end{pmatrix}$$

$H \nearrow$ $\quad H$

$$|-,W\rangle = \sin\theta/2\ e^{-i\varphi}|+,z\rangle - \cos\theta/2\ |-,z\rangle$$

$$\langle W,-| = \sin\theta/2\ e^{i\varphi}\langle z,+| - \cos\theta/2\ \langle z,-|$$

Suppose that at $t=0$

$$|\psi, 0\rangle = |+, W\rangle$$

If at $t=0$

$|\psi,0\rangle = |+,H\rangle$

$\text{Prob}(|+,H\rangle \to |-,H\rangle) = 0$

$|\psi,t\rangle = e^{-i/\hbar E_+ t}|+,H\rangle$

What is the probability at at time t the system is found in the state $|-,W\rangle$?

$$\text{Prob}(+W \to -W) = \left|\langle -,W|\psi,t\rangle\right|^2$$

$$|\psi,t\rangle = e^{-i/\hbar \hat{H} t}\,|\psi,0\rangle$$

$$= e^{-i/\hbar \hat{H} t}\,|+,W\rangle$$

$$= e^{-i/\hbar \hat{H} t}\left[\cos\theta/2\ |+,z\rangle + \sin\theta/2\ e^{+i\varphi}|-,z\rangle\right]$$

$$= \cos\theta/2\ e^{-i\frac{E_+ t}{\hbar}}|+,z\rangle + \sin\theta/2\ e^{+i\varphi}e^{-i\frac{E_- t}{\hbar}}|-,z\rangle$$

$$= e^{-iE_+ t/\hbar}\left[\cos\theta/2\ |+,z\rangle + \sin\theta/2\ e^{-i/\hbar(E_+ - E_-)t}e^{i\varphi}|-,z\rangle\right]$$

phase is Not measurable

$$\text{Prob}(|+,W\rangle \to |-,W\rangle) = \left|\langle -,W|\psi,t\rangle\right|^2$$

$$= \Big|\left(\sin\theta/2\ e^{i\varphi}\langle z,+| - \cos\theta/2\ \langle z,-|\right)$$

$$\left(\cos\theta/2\ |+,z\rangle + \sin\tfrac{\theta}{2}\,e^{i\varphi}e^{-i\Delta\omega t}|-,z\rangle\right)\Big|^2$$

$$= \left|\sin\theta/2\ \cos\theta/2\ \cancel{e^{i\varphi}} - \sin\theta/2\ \cos\theta/2\ e^{i\Delta\omega t}\ \cancel{e^{i\varphi}}\right|^2$$

$$= \left| S\theta/_2 \, Co\theta/_2 \, \cancel{e^{i\phi}} - S\theta/_2 \, Co\theta/_2 \, e^{i\Delta\omega t} \cancel{e^{i\phi}} \right|^2$$

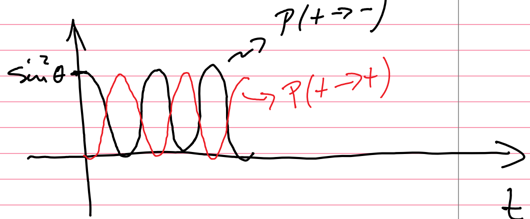
$$= S^2\theta/_2 \, Co^2\theta/_2 \left| 1 - e^{i\Delta\omega t} \right|^2 ; \quad \Delta\omega = \frac{E_+ - E_-}{2}$$

$$= \frac{1}{4} \sin^2\theta \left[1 + 1 - e^{i\Delta\omega t} - e^{-i\omega t} \right]$$

$$= \frac{\sin^2\theta}{2} \left(1 + \cos\Delta\omega t \right) \leq 1 \checkmark$$

$$P(+\to -) = S^2\theta \, \cos^2 \frac{\Delta\omega t}{2}$$

the Probability for flipping varies sinusoidally



$$P(+\to +) = |\langle \omega, + | \psi, + \rangle|^2$$
$$= 1 - P(+\to -)$$

Example: Neutrino Oscillations : $\hat{H} = \hat{H}_0 + \hat{H}_{int}$

Kinetic Energy

$\hat{H}_0 \uparrow \hat{H}$; $\hat{H}_{int} \hookrightarrow \hat{H}_{weak} \uparrow \hat{W}$

$$\left.\begin{aligned} \pi^+ &\to e^+ \nu_e \\ &\to \mu^+ \nu_\mu \\ &\to \tau^+ \nu_\tau \end{aligned}\right\}$$

$\nu_e \longrightarrow$ [detector] ; # ; # detected

$\hat{H}_{weak} |\nu_e\rangle = \pm |\nu_e\rangle$

$t = 0$ $\quad |\psi, 0\rangle = |\nu_e\rangle \text{ "=" } |+, \omega\rangle$

$$|\nu_e\rangle = a|+, z\rangle + b|-, z\rangle$$

$$E_\pm = E_0 \pm A \qquad \left.\begin{aligned} E_+ &= \sqrt{P_+^2 + m_+^2} \\ E_- &= \sqrt{P_-^2 + m_-^2} \end{aligned}\right\} \quad c = 1 \qquad E^2 - c^2\vec{P}^2 = m^2c^4$$

$$E_\pm \approx P_\pm \left(1 + \frac{m_\pm^2}{2P_\pm^2} \right) \Rightarrow \Delta\omega = \frac{E_+ - E_-}{\hbar} = \frac{m_+^2 - m_-^2}{2P\,\hbar}$$

$$P(+ \xrightarrow{\nu_e \to \nu_\mu} -) = \sin^2\theta \, S^2\left(\frac{\Delta m^2 \, t}{4P} \right) \qquad \hbar = 1$$

$$\Delta m^2 = m_+^2 - m_-^2$$

$$= \sin^2\theta \, \sin^2\left(\frac{\Delta m^2 t}{4E}\right)$$

The distance that the neutrino travels in time t:

$$L = vt \approx ct \qquad t = L/c$$

$$P(\nu_e \to \nu_\mu) = \sin^2\theta \, \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$\frac{\Delta m^2 L}{4E}\left(\frac{c^4}{\hbar c}\right) = (1.27)\,\frac{\Delta m^2 L}{E}$$

E : in GeV

Δm^2 : eV

L : km

$$P(\nu_e \to \nu_\mu) = \sin^2\theta \, \sin^2\left(\frac{1.27\,\Delta m^2 L}{E}\right)$$