We found the eigenkets of $\hat{n}\cdot\vec{\sigma} = \begin{pmatrix} \cos\theta & s\theta\, e^{-i\varphi} \\ s\theta\, e^{+i\varphi} & -\cos\theta \end{pmatrix}$ ✓

$*$ $|+,\hat{n}\rangle = \begin{pmatrix} \cos\theta/2 \\ s\theta/2\, e^{i\varphi} \end{pmatrix}$ $\quad |-,\hat{n}\rangle = \begin{pmatrix} s\theta/2\, e^{-i\varphi} \\ -\cos\theta/2 \end{pmatrix}$ $*$ $*$

$\hat{n} = (s\theta\cos\varphi,\ s\theta\, s\varphi,\ \cos\theta)$

When $\hat{n} = \hat{z} = (0,0,1)$ $\quad \hat{n}\cdot\vec{\sigma} = \sigma_z$, $\quad |+,z\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓ $*$; $\quad |-,z\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ✓ $*$

when $\hat{n} = \hat{x} = (1,0,0)$; $\quad \hat{n}\cdot\vec{\sigma} = \sigma_x$; $\quad |+,x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\quad |-,x\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

when $\hat{n} = \hat{y} = (0,1,0)$ $\quad \hat{n}\cdot\vec{\sigma} = \sigma_y$; $\quad |+,y\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$; $\quad |-,y\rangle = \frac{-1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$

Note the $(\hat{n}\cdot\vec{\sigma})$ is the dot product of two vectors. $\qquad \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$

$\mathcal{U}(R)\left[(\hat{n}\cdot\vec{\sigma})\,|\pm,\hat{n}\rangle = \pm|\pm,\hat{n}\rangle\right] \qquad \mathcal{U}^\dagger(R) = \mathcal{U}^{-1}(R)$

$\mathcal{U}(R)\,(\hat{n}\cdot\vec{\sigma})\,\mathcal{U}^\dagger\ \mathcal{U}|\pm,\hat{n}\rangle = \pm\ \mathcal{U}|\pm,\hat{n}\rangle$

In particular, $\quad \sigma_3\,|\pm,z\rangle = \pm\,|\pm,z\rangle$

$\mathcal{U}(R)\,\sigma_3\,\mathcal{U}^\dagger(R)\ \mathcal{U}(R)\,|\pm,z\rangle = \pm\ \mathcal{U}(R)\,|\pm z\rangle$

$\mathcal{U}(R)\,\sigma_3\,\mathcal{U}^\dagger(R) = \mathcal{U}(R)\,\hat{z}\cdot\vec{\sigma}\,\mathcal{U}^\dagger(R)$

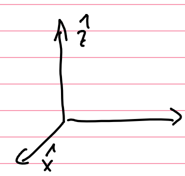
If R is rotation that takes $\hat{z}\to\hat{n}$

$\mathcal{U}(R)\,\hat{z}\cdot\vec{\sigma}\,\mathcal{U}^\dagger(R) = \hat{n}\cdot\vec{\sigma}$

& $\quad \mathcal{U}(R)\,|\pm,z\rangle = |\pm,\hat{n}\rangle$

It must be that,

$$\mathcal{U}(R) = \begin{pmatrix} \cos\theta/2 & s\theta/2\, e^{-i\varphi} \\ s\theta/2\, e^{i\varphi} & -\cos\theta/2 \end{pmatrix} \quad * *$$



Check, $\quad \mathcal{U}^\dagger\mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1 \qquad \begin{pmatrix} \cos\theta/2 & s\theta/2\, e^{-i\varphi} \\ s\theta/2\, e^{i\varphi} & -\cos\theta/2 \end{pmatrix}\begin{pmatrix} \cos\theta/2 & s\theta/2\, e^{-i\varphi} \\ s\theta/2\, e^{i\varphi} & -\cos\theta/2 \end{pmatrix}$

$= \begin{pmatrix} \cos^2\theta/2 + s^2\theta/2 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$ ✓

Also,

$\mathcal{U}(R)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & s\theta/2\, e^{-i\varphi} \\ s\theta/2\, e^{i\varphi} & -\cos\theta/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} - \\ - \end{pmatrix}$

$V_i \to V_i'$

T_{ij}

X_i

$\begin{pmatrix} - \\ - \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow |+,\hat{n}\rangle$

$|+,\hat{z}\rangle \xrightarrow{\mathcal{U}} |+,\hat{n}\rangle$

$\mathcal{U}\,\hat{z}\,\mathcal{U}^\dagger = \hat{n}$

$|\psi\rangle \xrightarrow{R} \mathcal{U}(R)|\psi\rangle$

$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{R} \begin{pmatrix} \cos\theta/2 & s\theta/2\, e^{-i\varphi} \\ s\theta/2\, e^{i\varphi} & -\cos\theta/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$

$\underbrace{\det\mathcal{U} = +1}_{\mathcal{U}}$

$\mathcal{U}(R(\theta',\varphi'))\,\mathcal{U}(R(\theta,\varphi))$ ✓ ✓

$= \mathcal{U}(R(\xi,\zeta))$ ✓

$$= \begin{pmatrix} \cos\theta/_2 \\ \sin\theta/_2\, e^{i\varphi} \end{pmatrix} \quad \checkmark$$

$$\mathcal{U}(R) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{U}^{\dagger}(R) = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{+i\varphi} & -\cos\theta \end{pmatrix}$$

$$\mathcal{U}(R(\hat{z} \to \hat{x})) = \mathcal{U}(R(\theta = \pi/_2, \varphi = 0)) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad ; \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

~~~ // ~~~ // ~~~

We can also write $\mathcal{U}(R)$ in the form:

$$\mathcal{U}(R) = e^{i \frac{\vec{\xi} \cdot \vec{\sigma}}{2}} = \begin{pmatrix} \cos\theta/_2 & \sin\theta/_2\, e^{-i\varphi} \\ \sin\theta/_2\, e^{i\varphi} & -\cos\theta/_2 \end{pmatrix}$$

$$= \sum_n (i)^2 \frac{1}{2^n} \frac{\xi^n}{n!} (\hat{\xi} \cdot \vec{\sigma})^n \qquad \vec{\xi} = \vec{\xi}(\theta, \varphi)$$

$$(\hat{\xi} \cdot \vec{\sigma})^0 = 1 \qquad (\hat{\xi} \cdot \vec{\sigma})^2 = \hat{\xi} \cdot \hat{\xi} = 1$$

$$\underline{(\hat{\xi} \cdot \vec{\sigma})^{2n} = 1} \qquad \underline{(\hat{\xi} \cdot \hat{n})^{2n+1} = (\hat{\xi} \cdot \hat{n})}$$
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