In quantum Mechanics all observables in the end are expressed in the form:

$$\text{Probs} = |\langle\psi|\varphi\rangle|^2$$

Any symmetry S must be such that

$$|\psi\rangle \xrightarrow{S} |\psi'\rangle = \hat{U}|\psi\rangle$$

$$|\varphi\rangle \xrightarrow{S} |\varphi'\rangle$$

$$|\langle\psi|\varphi\rangle| \xrightarrow{S} |\langle\psi'|\varphi'\rangle| = |\langle\psi|\varphi\rangle|$$

$$|\langle\psi|\varphi\rangle| = |\langle\varphi|\psi\rangle| \quad ***$$

this allows for two possibilities:

$$\langle\psi|\varphi\rangle \xrightarrow{S} \langle\psi|\varphi\rangle \qquad \text{"}U^{+} = U^{-1}\text{"} \; *$$

or

$$\langle\psi|\varphi\rangle \xrightarrow{S} \underbrace{\langle\varphi|\psi\rangle} \qquad \text{"}\hat{U}\text{ will be antiunitary"}$$

(Time reversal operator)

$$\langle\psi|\varphi\rangle \xrightarrow{S} \langle\psi'|\varphi'\rangle = \langle\psi|U^{+}U|\varphi\rangle$$

$$= \langle\psi|\varphi\rangle \Rightarrow U^{+}U = \mathbb{1} \; *$$

$$U^{+} = U^{-1}$$

$$UU^{+} = \mathbb{1}$$

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Groups Definition

Group is a set of elements : $\{g_1, g_2, \dots, g_N\} : G$ with the following four properties:

1) There $\exists$ a binary operation, called the group multiplication rule, that associates two group elements G with a third group element in G.

$$\underset{\substack{\uparrow \\ \text{binary operation}}}{g \cdot g'} = g'' \quad \ni \quad g, g', \& \, g'' \in G$$

Note that in general $g \cdot g' \neq g' \cdot g$

when $g \cdot g' = g' \cdot g$ the group is abelian

2) Associative property:

$$g_1 \cdot (\underbrace{g_2 \cdot g_3}_{g'}) = (\underbrace{g_1 \cdot g_2}_{g''}) \cdot g_3$$

3) Identity Element $e \ni \quad g \cdot e = e \cdot g = g$

$\forall \; g \in G$

Note: e is unique!

4) For group element $g \; \exists \; g' \quad \forall \, g, g' \in G$

$\ni \quad g \cdot g' = e \Rightarrow g' = g^{-1}$

$g \cdot g^{-1} = e \Rightarrow g^{-1} \cdot g = e$

Note: That g^{-1} is unique

Example 1) The of all integers is a group under addition.

Identity is Zero

Inverse of N is $-N$

This is abelian group $(\cdot) = (+)$

$N_1 \cdot N_2 = N_1 + N_2 = N_2 \cdot N_1 = N_2 + N_1$ ✓

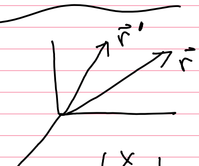
Abelian.

2) The set of all real numbers under multiplication,

Identity element: 1

For x the inverse $1/x$ *

We have to exclude Zero.



Rotations

$\vec{r} \xrightarrow{R} \vec{r}\,'$

$\underbrace{r_i \xrightarrow{R} r_i'}$

$r_i' = R_{ij} r_j$

$\vec{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\vec{r} \cdot \vec{r} = \vec{r}\,' \cdot \vec{r}\,' \Rightarrow R^T = R^{-1}$

"Orthogonal Matrix"

Identity Element $\vec{r} \longrightarrow \vec{r}$

$\Rightarrow$ It is represented by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_{3\times3}$

$R^T R = \mathbb{1} \qquad \det(R^T R) = 1$

$\det R^T \det R = 1 \Rightarrow (\det R)^2 = 1$

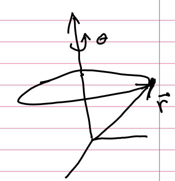
$\Rightarrow \det R = \pm 1$ (For (-1) case can be obtained by combining rotations w/ reflections)

We only need to focus on $\boxed{\det R = 1}$

Note: For $\det R = -1$; $\det(R_1 R_2) \neq -1$

In 3-d space R is characterized three continuous parameters

$$\Rightarrow \boxed{R_{ij} \approx \delta_{ij} + \omega_{ij}} \; *$$ small in some sense



$$R^T R = 1 \Rightarrow (R^T)_{il}(R)_{lj} = \delta_{ij}$$

$$R_{li} R_{lj} = \delta_{ij}$$

$$(\delta_{li} + \omega_{li})(\delta_{lj} + \omega_{lj}) = \delta_{ij}$$

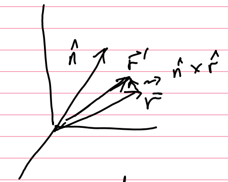
$$\delta_{ij} + \omega_{ij} + \omega_{ji} + O(\omega^2) = \delta_{ij}$$

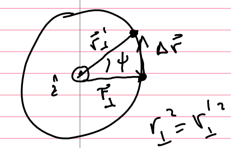
$$\boxed{\omega_{ij} = -\omega_{ji}} \; *$$

$$\Rightarrow \omega_{11} = -\omega_{11} = 0$$

$$\begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & - & 0 \end{pmatrix} \Rightarrow 3 \text{ indep. parameters}$$

$$\Delta\vec r = \hat z \times \vec r$$
$$\vec r\,' = \vec r + \hat z \times \vec r\,(a)$$
$$r'^2 = r^2$$





$$\vec r = \vec r_\perp + \vec r_\parallel$$
$$\vec r\,' = \vec r\,'_\perp + \vec r_\parallel$$
$$r_\perp^2 = r_\perp'^2$$

$$\vec r\,' = \vec r + \hat n \times \hat r\, \xi \quad \text{infinitesimal}$$

$$\vec r\,' \cdot \vec r\,' = r^2$$

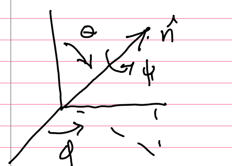
$$\hat n = \hat n(\theta, \varphi) \; *$$
$$a = a(\psi) \; *$$

$$\vec r\,' \cdot \vec r\,' = r^2 + a^2 (\hat n \times \hat r)^2$$
$$|\hat n \times \hat r| = \sim \xi$$
$$\xi = \xi(\theta, \varphi,$$

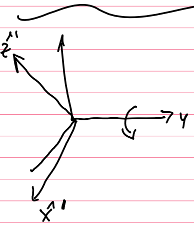
$$R_{ij} = \delta_{ij} + \epsilon_{ijk}\psi_k$$
$$\psi_k = (0, 0, \psi)$$

In general any $N \times N$ antisymmetric matrix has $\left(\frac{N(N-1)}{2}\right)$ indep. parameters

Likewise any $N \times N$ symmetric matrix has $\frac{N(N+1)}{2}$ indep. parameters



$$R = R(\theta, \varphi, \psi) = R_{\hat n}(\psi) = R(\hat n \psi) = R(\vec \psi)$$



The Ket Vectors under Rotations

$$|\psi\rangle \xrightarrow{R} |\psi'\rangle$$

$$|\psi'\rangle = \hat U(R)|\psi\rangle \qquad U^\dagger = U$$

$$R = \mathbb{1}; \quad U = \hat{\mathbb{1}}$$

For an infinitesimal Rotation: $R_{ij} = \delta_{ij} + \omega_{ij}$

$$\boxed{U(\delta_{ij} + \omega_{ij}) = \hat{\mathbb{1}} + \left(\frac{i}{2\hbar}\right)\omega_{ij}\hat J_{ij}}$$ Red; convention

$$U^\dagger U = 1 = \left(1 - \frac{i}{2\hbar}\omega_{ij}J^\dagger_{ij} + \ldots\right)\left(1 + \frac{i}{2\hbar}\omega_{ij}J_{ij} + \ldots\right)$$

$$= 1 + \frac{i}{2\hbar}\left(\omega_{ij}\hat{J}_{ij} - \omega_{ij} J_{ij}^{+}\right) + \dots$$

$$= \mathbb{1}$$

$$\Rightarrow \quad \omega_{ij}\hat{J}_{ij} = \omega_{ij}\hat{J}_{ij}^{+} \quad \Rightarrow \quad \underline{\hat{J}_{ij} = \hat{J}_{ij}^{+}}$$

Since $\omega_{ij} = -\omega_{ji} \quad \Rightarrow \quad \hat{J}_{ij} = -\hat{J}_{ji}$

$\Rightarrow$ 3 indep. operators

$$J_{12}^{+} = J_{12} \; ; \; J_{14}^{+} = J_{14}$$
$$J_{13}^{+} = J_{13}$$

Since, $\underline{R_1 R_2} = R \Rightarrow \boxed{U(R_1)\,U(R_2) = U(R) = U(R_1 R_2)}$ *

Homomorphism

$O(3) \sim SU(2)$

This implies conditions on $\hat{J}_{ij}$

Look at

$R_\omega = \mathbb{1} + \omega$

$$U(R'^{-1})\,\underbrace{U(R_\omega)}\,U(R') = U(R'^{-1} R_\omega R')$$

$$U(R'^{-1})\left(\mathbb{1} + \frac{i}{2\hbar}\omega_{ij}\hat{J}_{ij}\right)U(\underline{R'}) = \mathbb{1} + \frac{i}{2\hbar}\underbrace{(R'^{-1}R_\omega R')_{lm}}_{\omega'_{lm}} J_{lm}$$

$$\cancel{\frac{i}{2\hbar}}\,\omega_{ij}\,U(R'^{-1})\,\hat{J}_{ij}\,U(R') = \cancel{\frac{i}{2\hbar}}\,(R'^{-1}R_\omega R')_{lm} J_{lm}$$

$$U(R'^{-1}) = U^{+}(R') \qquad = \omega_{nk} R'_{nl} R'_{km} J_{lm}$$

$$\checkmark \quad \boxed{U^{+}(R')\,\hat{J}_{ij}\,U(R') = R'_{nl}R'_{km}\hat{J}_{lm}} \quad \checkmark$$

Recall that under Rotations any Vector

$$V_i' = R_{ij} V_j$$

$$T_{ij}' = R_{il} R_{jm} T_{lm}$$

$$U(R') = \mathbb{1} + \frac{i}{2\hbar}\omega'_{lm}\hat{J}_{lm} \qquad R_{nl} = \delta_{nl} + \omega'_{nl}$$

$$U^{+}(R') = \mathbb{1} - \frac{i}{2\hbar}\omega'_{lm}\hat{J}_{lm}$$

$$\Rightarrow \quad -\frac{i}{2\hbar}\,\omega'_{lm}\left[\hat{J}_{lm}, \hat{J}_{nk}\right] = \omega_{km}\hat{J}_{nm} + \omega_{nl}\hat{J}_{lk}$$

$$= \underbrace{\omega_{lm}}_{-\omega_{ml}}\underbrace{\left[\delta_{kl}\hat{J}_{nm} - \delta_{nm}\hat{J}_{lk}\right]}_{\text{only anti-symm. contributes}}$$

$$[J_{lm}, J_{nk}] = \frac{\omega_{em}}{2}\left[\delta_{kl} J_{nm} - \delta_{nm} J_{lk} - (\delta_{km} \hat{J}_{nl} - \delta_{nl} \hat{J}_{mk})\right]$$

$$[\hat{J}_{lm}, \hat{J}_{nk}] = i\hbar\left[\delta_{kl} \hat{J}_{nm} - \delta_{nm} \hat{J}_{lk} - \delta_{km} \hat{J}_{nl} + \delta_{nl} \hat{J}_{mk}\right]$$

$\hat{J}_{ik}$

$$U = 1 + \frac{i}{2\hbar} \omega_{ij} \hat{J}_{ij}$$

$\hookrightarrow$ generators of the Rotation

$$\hat{J}_{lm} = -\hat{J}_{ml} \Rightarrow \hat{J}_{lm} = \epsilon_{lmi} J_i$$

$$[\hat{J}_i, \hat{J}_k] = i\hbar \epsilon_{ikl} \hat{J}_l$$

generators

$$U|\psi\rangle = |\psi'\rangle$$

$\Rightarrow$ The complete set of operators must contain $\{\hat{H}, J^2, J_3, \ldots\}$

$\Rightarrow$ States are characterized by the eigenvalues $\{E, j(j+1), m, \ldots\}$

$$U = 1 + \frac{i}{2\hbar} \omega_{ij} \hat{J}_{ij} + \ldots$$

$$\simeq 1 + \frac{i}{2\hbar} \omega_{ij} \epsilon_{ijk} J_k$$

$$= 1 + \frac{i}{\hbar} \delta\vec{\theta} \cdot \vec{J}$$

infinitesimal

$$\epsilon_{nji} \theta_i = \frac{1}{2} \epsilon_{nji} \epsilon_{ilm} \omega_{lm}$$

$$\theta_i = \frac{1}{2} \epsilon_{ilm} \omega_{lm}$$

$$\vec{\theta} \equiv (\delta\vec{\theta}) N \qquad \delta\vec{\theta} = \left(\frac{\vec{\theta}}{N}\right)$$

$$U(R(\vec{\theta})) = \lim_{N \to \infty} \left(1 + \frac{i}{\hbar} \frac{\vec{\theta} \cdot \vec{J}}{N}\right)^N$$

$$[\hat{\theta} \cdot \hat{J}, \hat{\theta} \cdot \hat{J}] = 0$$

$$e^x = \lim_{N \to \infty} \left(1 + \frac{x}{N}\right)^N$$

for finite Rotations $U(R(\theta)) = e^{i/\hbar\, \vec{\theta} \cdot \vec{J}}$