Elements of Group Theory

The of Linear Transformations forms a group.

$$\{a, b, c, \cdots\} \rightarrow \{T(a), T(b), \cdots\}$$

$$\ni \left.\begin{array}{l} T(a+b) = T(a) + T(b) \\ T(\alpha a) = \alpha T(a) \end{array}\right\}$$

Example: In 3-d Vector space $(X_1, X_2, X_3) \xrightarrow{LT} (X_1', X_2', X_3')$

$$X_i' = M_{ij} X_j \Rightarrow X_i' + Y_i' = M_{ij}(X_j + Y_j)$$

$$T(\vec{x} + \vec{y}) = M_{ij}(X_j + X_j)$$

$$= M_{ij} X_j + M_{ij} X_j = X_i' + Y_j'$$

$$= T(\vec{X}) + T(\vec{Y})$$

$$T(\alpha \vec{X}) = \alpha T(\vec{X})$$

Spse we have to linear transf. T_1 & T_2

$$\vec{X} \xrightarrow{T_1} \vec{X}' \xrightarrow{T_2} \vec{X}'' \qquad \vec{X}' = T_1(\vec{x})$$

$$\vec{X} \xrightarrow{T_2 T_1} \vec{X}'' \qquad \vec{X}'' = T_2(\vec{X}') = T_2(T_1(\vec{X}))$$

$$\vec{X}'' = \underbrace{T_2 T_1}_{T \equiv T_2 T_1}(\vec{X}) \qquad \vec{X} \longrightarrow \vec{X}$$

$\therefore$ The set of Linear Transf. forms a group.

$$T(\vec{X}) = \vec{X}'$$

If, $X_i' = M_{ij} \tilde{X}_j$ ✓ $GL(3)$

Then we say that the set of matrices M forms a representation of the Linear group.

Defn Homomorphism: given two groups $G : \{g_1, g_2, \cdots\}$
$G' : \{g_1', g_2', \cdots\}$

If $\exists$ a mapping $\ni g : G \qquad g \to g'$

$$\ni \quad g_1 \cdot g_2 = g_3 \implies g_1' \cdot g_2' = g_3'$$

Product Rules are in general different

If the mapping is 1-1 the Homomorphism is an Isomorphism.

For example: If $|\psi\rangle \xrightarrow{T_1} |\psi'\rangle \xrightarrow{T_2} |\psi''\rangle$

$$|\psi\rangle \xrightarrow{T_2 T_1} |\psi''\rangle$$

$$U(T)|\psi\rangle = |\psi'\rangle \qquad T \to U(T) \to$$

$$T_2 T_1 \longrightarrow U(T_2)\, U(T_1)$$

Homomorphism

Definition of a Representation: A representation of a group G is a homomorphism of G into the group of Linear Transformations.

Rep: $G \longrightarrow GL(n)$

$$\{g_1, g_2, \cdots\} \longrightarrow \{GL(g_1), GL(g_2), \cdots\}$$

Small

$$|\psi'\rangle = \underline{U}(T)\, |\psi\rangle$$

(n) *

$|\psi'\rangle = U(T)|\psi\rangle$

For an n-state system : $|\psi\rangle = \left\{ \sum_{i=1}^{n} c_i |e_i\rangle \right\}^{*}$

$\Rightarrow$ "Finite Dimensional Reps"

$U^{+} = U^{-1}$

$U(T)|e_i\rangle = |e_i'\rangle$

$\mathbb{1} = |e_l\rangle\langle e_l|$

$|e_l\rangle \underbrace{\langle e_l|U(T)|e_i\rangle} = |e_i'\rangle$

$|e_i'\rangle = D_{li}(T)|e_l\rangle$

$\hookrightarrow$ A linear Rep. of $U(T)$ of dimension $\underline{n}$

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$|e_i\rangle \xrightarrow{T_1} |e_i'\rangle \xrightarrow{T_2} |e_i''\rangle \Rightarrow |e_i\rangle \xrightarrow{T_2 T_1} |e_i''\rangle$

$|e_i''\rangle = D_{il}(T)|e_l'\rangle = D_{il}(T_2) D_{lk}(T_1)|e_k\rangle$

$|e_i''\rangle = D_{il}(T_2) D_{lk}(T_1)|e_k\rangle$

$|e_i''\rangle = D_{ik}(T_2 T_1)|e_k\rangle$

** $\Rightarrow$ $\underbrace{D_{ik}(T_2 T_1)} = \underbrace{D_{il}(T_2) D_{lk}(T_1)}$

"D are n×n matrices"

If $\underset{\sim}{D}$ is a rep. of G

$\Rightarrow D(g_1)D(g_2) = D(g_1 g_2)$

$\underbrace{g_1 \cdot g_2 = g_3} \longrightarrow \underbrace{D(g_1)D(g_2) = D(g_1 g_2)}$

$|\psi'\rangle = U(T)|\psi\rangle \qquad |\psi\rangle = c_1|e_1\rangle + c_2|e_2\rangle$

$$|\psi'\rangle = U(T)|\psi\rangle \qquad |\psi\rangle = c_1|e_1\rangle + c_2|e_2\rangle$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \xrightarrow{R} \underbrace{\begin{pmatrix} - & - \\ - & - \end{pmatrix}}_{D_{2\times 2}} \begin{pmatrix} a \\ b \end{pmatrix}$$

If $|\psi\rangle = \sum_{i=1}^{n} c_i |e_i\rangle$

$$\underset{\sim}{D}(R) : (n \times n \text{ matrix})$$

Spse we two Reps of T : $\underbrace{D_1(T)}_{n\text{-dim}}$ & $\underbrace{D_2(T)}_{m\text{-dim}}$

$$D_1(T_1 T_2) = D_1(T_1) D_1(T_2)$$

$$\underbrace{D(T)}_{(n+m)\text{ dim. rep. of } T} = \begin{pmatrix} D_1(T) & \underset{\sim}{\phi} \\ \underset{\sim}{\phi} & D_2(T) \end{pmatrix}_{(n+m)\times(n+m)}$$

$$\Rightarrow D(T_1 T_2) = D(T_1) D(T_2)$$

$$D(T) \underbrace{\begin{pmatrix} (\vec{V}_n) \text{ "n"} \\ (\vec{W}_m) \text{ "m"} \end{pmatrix}}_{(n+m)\text{ dim vector}} = \begin{pmatrix} D_1(T) \begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{pmatrix} \\ D_2(T) \begin{pmatrix} W_1 \\ \vdots \\ W_m \end{pmatrix} \end{pmatrix}$$

Exmple: the Y's $\qquad Y_l^m \xrightarrow{R} \sum_{m'=-l}^{l} C_{m'}^{(l)} Y_l^{m'}$

(l fixed)

the $Y_l^{m'}$'s form the $2l+1$ reps. of $\underline{R}$.

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Exmple: The group G: (Real #'s under Addition)

$$\begin{cases} x + y = z \checkmark & x + 0 = x \checkmark \\ & x - x = 0 \checkmark \\ x + (y+z) = (x+y) + z & \end{cases}$$

The 2-dim rep. has the form:

$$x \longrightarrow T_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad x \in \mathbb{R}$$

$$T_x T_y = \begin{pmatrix} 1 & x+y \\ & \end{pmatrix} = T_{(x+y)}$$

$$\begin{cases} \underline{T_x T_y} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = T_{(x+y)} \\ T_x^{-1} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} ; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

The dimension of this rep is $\underline{2}$

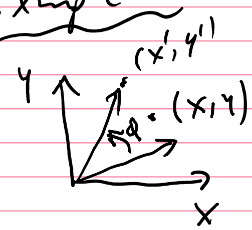
$$T_x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + x u_2 \\ u_2 \end{pmatrix}$$

$$T_x \begin{pmatrix} u_1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} \quad \text{i.e. "Invariant"}$$

$$T_x \begin{pmatrix} 0 \\ u_2 \end{pmatrix} = \begin{pmatrix} x u_2 \\ u_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ u_2 \end{pmatrix}$$

All vectors of the form: $\begin{pmatrix} u_1 \\ 0 \end{pmatrix}$ comprise an invariant subspace.

Example 2: G_{R_2}: { set of rotations in 2-d space } $\equiv O(2)$



$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{R} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$D^T = D^{-1}$ ↗ 2 dim.

$\det D = 1$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}}_{2\times 2 \text{ matrices}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{|e_1\rangle} + y \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{|e_2\rangle}$$

$$D(\theta) D(\varphi) = D(\theta + \varphi)$$

$$= D(\varphi + \theta) = D(\varphi) D(\theta) \quad \text{"Abelian"}$$

Defn:

$$|\pm\rangle = \frac{1}{\sqrt{2}} \left(\mp |e_1\rangle + i |e_2\rangle \right)$$

$$D(\varphi) |+\rangle = e^{-i\varphi} |+\rangle$$

$$D(\varphi) |-\rangle = e^{i\varphi} |-\rangle$$

$\therefore$ $D(\varphi)$ in the $|\pm\rangle$ basis is diagonal.

$$(*)\quad D(\varphi) = \left(\begin{array}{c|c} e^{-i\varphi} & 0 \\ \hline 0 & e^{i\varphi} \end{array}\right)$$

$\Rightarrow$ The 2-dimensional Rep. of $SO(2)$ is reducible!

$e^{i\varphi}$: Reps. of $U(1)$: $u|\psi\rangle = |\psi'\rangle$

$$u|\psi\rangle = e^{i\varphi}|\psi\rangle$$

Theorem: Schurs Lemma

If $D(g)$ is an irred. rep. of G on a vector space V

& $\hat{A}$ is an arbitrary operator on V

Then If $D(g)\hat{A} = \hat{A}D(g)$ $\quad \forall g \in G$

then $\Rightarrow$ $\hat{A} = \lambda \mathbb{1}$ $\Rightarrow$ $\hat{A}$ has only one eigenvalue

Exmple:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\epsilon_{ijk}\hat{J}_k$$

$$\left.\begin{array}{l} U(R(\hat{n}\varphi)) = e^{-i/\hbar\, \hat{n}\cdot\vec{J}\varphi} \end{array}\right\}$$

$$U J_i U^{+} = R_{ij} J_j$$

$$\underbrace{\hat{n}\varphi}$$

$$[\hat{J}^2, J_i] = 0$$

$$U\hat{J}^2 U^{+} = \hat{J}^2$$

$$U\vec{J}^2 = \vec{J}^2 U \Rightarrow \underset{\sim}{D}\,\underset{\sim}{\hat{J}}^2 = \underset{\sim}{\hat{J}}^2\underset{\sim}{\hat{D}}$$

This is true for arbitrary $\hat{n}\varphi$, i.e.

$$\underset{\sim}{D}\,\underset{\sim}{\hat{J}}^2 = \underset{\sim}{\hat{J}}^2\underset{\sim}{\hat{D}} \quad \text{for all } g \in \mathbb{R}$$

$$\underset{\sim}{D}\,\underset{\sim}{\hat{J}}^2 = \underset{\sim}{\hat{J}}^2\,\underset{\sim}{D} \quad \text{for all } g \in \mathbb{R}$$

$$\underset{\sim}{\hat{J}}^2 = \lambda\,\mathbb{1} \qquad \lambda = \hbar^2 j(j+1)$$

The Basis Vectors : $|j, m\rangle$ *

$$J^2 |j, m\rangle = \hbar^2 \underbrace{j(j+1)}\, |j, m\rangle$$

$$J_z |j, m\rangle = \hbar m\, |j, m\rangle$$

$$\left(\underset{\sim}{D}(R)\right)_{m'm} = \langle j\, m' |\, U(R)\, | j\, m\rangle$$

$(2j+1)\times(2j+1)$

If $\underset{\sim}{A}\,\underset{\sim}{D}(g) = \underset{\sim}{D}(g)\,\underset{\sim}{A} \Rightarrow \underset{\sim}{A}$ "Casimir Operator"