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On Accidental Degeneracy in Classical and Quantum Mechanics

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The theory of accidental degeneracy is surveyed, particular attention being paid to the connection between the accidental degeneracy of the two-dimensional isotropic harmonic oscillator and the theory of angular momentum.

ONE of the reasons for introducing group theory into the study of quantum mechanics is that some of the degeneracy of many quantum mechanical problems may be accounted for as a forced degeneracy that is due to some symmetry possessed by the system. For instance, the spherical symmetry of the central force problems has as a consequence the conservation of angular momentum; in addition there are many states, possessing different values of the z component of angular momentum, which have the same total angular momentum and the same energy. The forced degeneracy is a consequence of Schur's lemma, which restricts the form of a Hamiltonian that is invariant under all the transformations of a symmetry group.

Many problems have an obvious geometrical symmetry, such as the spherical symmetry of a central force, the crystallographic symmetry of a solid body, or the polyhedral symmetry of a molecular system. Yet in a great many of the textbook examples, even after the degeneracy due to this symmetry has been identified and explained, it will be found that there are some levels which have no need to be degenerate due to the symmetry but which nevertheless possess the same energy. This is the so-called "accidental" degeneracy. It is accidental in the sense that although Schur's lemma tells when certain eigenvalues of the Hamiltonian are necessarily equal, there is no reason that different sets of these eigenvalues cannot be equal to one another.

Even though accidents of this nature occur, it is nevertheless tempting to think that there may be present a higher symmetry which has been overlooked, and that the symmetry group which expresses this point of view will completely explain all the degeneracies present. As Alliluev points out in a recent paper,[1] such hidden symmetry actually exists in a number of familiar instances—the Kepler problem and the isotropic harmonic oscillator, for instance. In a sense, however, the problem is not so much to find a larger group, for it is easy to imagine the group of all transformations which leave a Hamiltonian invariant, and even to hope that it will be a Lie group. A more serious problem is to identify the group with something which possesses physical significance—its generators with constants of the motion, for instance.

Instances of accidental degeneracy are abundant in the soluble problems of quantum mechanics. One of the best known is the Kepler problem, or, in its more usual quantum mechanical formulation, the hydrogen atom. A particle moves in three-dimensional space subject to the influence of an attractive potential which varies inversely as the radius, $1/r$. If the states are cataloged according to the three quantum numbers m, l, and n, where l is the angular momentum, m the z component of the angular momentum, and n is the total quantum number, it is well known that the energy depends only upon the total quantum number n and not upon l or m. However, the spherical symmetry of the potential (and kinetic energy) would only require a degeneracy in m, and not in l as well. In fact, there are n^2 states, with l values ranging from 0 to $n-1$; and m values from $-l$ to $+l$, rather than the $(2l+1)$ degenerate states which we would expect from the conservation of angular momentum alone.

Another prominent example, or rather class of examples, is composed of isotropic harmonic oscillators of various dimensionalities. In two dimensions, for instance, the potential has

[1] S. P. Alliluev, Soviet Phys. JETP **6**, 156–159 (1958).

cylindrical symmetry, and thus the Hamiltonian is invariant under the two-dimensional rotation group, which has at most two-dimensional irreducible representations. Nevertheless, it is found that the energy is proportional to $(m+n+1)$, where m and n are the quantum numbers arising from the separation of the harmonic oscillator equation in Cartesian coordinates. Thus, the multiplicity of a level is equal to the number of ways that an integer N may be written as the sum of two non-negative integers; this number is $N+1$, which stands in marked contrast to the twofold degeneracy which was expected. Similar anomalies exist in higher dimensions, for which the harmonic oscillator is always much more highly degenerate than its spherical symmetry would indicate.

Other problems exhibiting a high degree of accidental degeneracy are the free particle enclosed by an impenetrable cubic box, and the quantum mechanical rigid rotor. The spherical rotor has $(2j+1)^2$-fold degeneracy rather than the $(2j+1)$-fold degeneracy which would be expected by virtue of its spherical symmetry; the symmetric rotor has $(2j+1)$ or $2(2j+1)$-fold degeneracy rather than the 1- or 2-fold degeneracy known to be caused by its cylindrical symmetry, and even the asymmetric rotor has the degeneracy of a sphere. The motion of a particle in a homogeneous magnetic field is another example of an excessively degenerate problem, although this degeneracy may be made more obvious by enclosing the particle in an impenetrable box so that the symmetry of the Euclidean group is lost. The Kepler problem in non-Euclidean space also yields some interesting results, as does the relativistic Kepler problem.

The fact that these accidental degeneracies are connected with the existence of constants of the motion in their classical analogues seems to have been clearly recognized by Pauli in 1926.[2] The constants of the motion may generate a Lie group which is larger than the group of obvious geometric symmetries of the configuration space of the problem. He made use of the stability of the semiaxes of the Keplerian orbits to employ constants of the motion in addition to the components of angular momentum in his discussion of the hydrogen spectrum. Actually these new constants date far back into the history of the Kepler problem. They are the components of a vector which points along the semimajor axis of the orbit, and whose length is equal to the eccentricity, and which depends only upon the semimajor axis. This vector establishes the direction of the orbit, just as the angular momentum determines its plane.

Although Pauli worked out the commutation rules for the components of angular momentum and of this new vector, called the Lenz vector,[3] it was apparently Klein who recognized them as the commutation rules of the four-dimensional rotation group.[4] Fock, by writing an integral equation for the hydrogenic wave functions in momentum space, was able to recognize the kernel of the equation as the Jacobian of a stereographic projection, and thereby by a transformation of variables to change the Schroedinger equation for the hydrogen atom into Laplace's equation for hyperspherical surface harmonics.[5] This made the role of the four dimensional rotation as a symmetry group—for the bound states, at least—of the Kepler problem quite obvious. For the unbound states, the symmetry group was the Lorentz group; for the parabolic orbits, the Euclidean group in three dimensions. In a commentary on Fock's paper, Bargmann showed explicitly that Fock's group is generated by the constants of the motion of the Kepler problem.[6]

Laporte reported on work of his own on this problem at a meeting of the American Physical Society in 1936,[7] and in a paper by Laporte and Rainich,[8] discusses a mathematical theory of stereographic projections, and the hidden symmetry groups which they may involve. Säenz, a student of Laporte's, wrote a thesis in 1949 in which he developed the idea of the stereographic parameters and applied it to the symmetric rotor, the Kepler problem, and the harmonic oscillator.[9] The idea was to find those problems whose Hamilton-Jacobi equations, under suitable trans-

[2] W. Pauli, Jr., Z. Physik **36**, 336–363 (1926).

[3] W. Lenz, Z. Physik **24**, 197–207 (1924).

[4] See L. Hulthen, Z. Physik **86**, 21–23 (1933).

[5] V. Fock, Z. Physik **98**, 145–154 (1935).

[6] V. Bargmann, Z. Physik **99**, 576–582 (1936).

[7] Otto Laporte, Phys. Rev. **50**, 400(A) (1936).

[8] Otto Laporte and G. Y. Rainich, Trans. Am. Math. Soc. **39**, 154–182 (1936).

[9] A. W. Säenz, "On integrals of the motion of the runge type in classical and quantum mechanics," Ph.D. thesis, University of Michigan (1949).

formation, became identical to the corresponding equation for force-free motion on a hypersphere of some dimensionality. He also attempted to account for the excess degeneracy in the Dirac-Kepler problem, but showed that a similar explanation would not work for the accidental degeneracy in that case; that is, it could not be reduced to force-free motion on a hypersphere.

The occurrence of accidental degeneracy in a large class of problems seems to be connected with the existence of bounded closed orbits in the analogous classical problem, and the destruction of the Keplerian constants in relativistic motion can be traced to the precession of the perihelion, by which the directional stability of the orbit is lost.

Hill, at the University of Minnesota, has been interested in the accidental degeneracy problem, and one of his students, J. M. Jauch, discussed the harmonic oscillator at some length in his thesis.[10] They discuss the two-dimensional Kepler problem, and find that its bound states, like those of the two-dimensional harmonic oscillator, are degenerate under the three-dimensional rotation group. They find a two-dimensional analogue of the Balmer formula, which has been found in the general case by Alliluev.[1] The fact that the half-integral representations of the rotation group occur with the harmonic oscillator attracted their notice and they remark that it is interesting that this is the first time that half-integral quantum numbers have occurred in connection with a problem involving nonrelativistic quantum mechanics. Another problem which interested them was the anisotropic harmonic oscillator, which still seems to have the three-dimensional rotation group as a symmetry group when the two frequencies are commensurable.

Hill has prepared a very interesting set of notes in which he discusses the philosophy of quantum mechanics at some length and gives a particularly nice survey of the accidental degeneracy problem.[11]

Perhaps one of the most interesting aspects of the theory of the two-dimensional harmonic oscillator has been the use made of it by Schwinger to develop the theory of angular momentum,[12] by exploiting the appearance of the three-dimensional rotation group as the symmetry group of the plane isotropic harmonic oscillator. In Schwinger's paper the language is that of second-quantized field theory, and of course no connection is made with the theory of accidental degeneracy. Nevertheless the two theories are intimately related, and a very far-reaching theory of angular momentum may be created by the use of the properties of the harmonic oscillator.

Schwinger's operator techniques are used very nicely in a paper by Johnson and Lippmann,[13] in which the problem of nonrelativistic cyclotron motion is treated and which involves the properties of the harmonic oscillator in its solution. In another paper they discuss the relativistic motion,[14] and in an abstract of a paper presented at a meeting they incidentally discover a constant of the motion which accounts for the accidental degeneracy in the Dirac-Kepler problem.[15]

In fact, Schwinger has done a very interesting job of reducing the radial equation of the Kepler problem to the radial equation of the harmonic oscillator.[16] This reduction depends upon the known fact that the substitution $\rho = r^2$ will convert Hermite functions into Laguerre functions, and thus will interconvert the two radial equations in question.

Perhaps the deepest connections of all between all these theories are to be found in the theory of the properties of the three-dimensional rotation group. The connection with the theory of the rotation group comes about in roughly the following fashion, and is a unique property of the three-dimensional rotation group. When the rotation group is parameterized, the param-

[10] J. M. Jauch, Phys. Rev. **55**, 1132(A) (1939); J. M. Jauch and E. L. Hill, Phys. Rev. **57**, 641–645 (1940); J. M. Jauch, "On contact transformations and group theory in quantum mechanical problems," Ph.D. thesis, University of Minnesota (1939).

[11] E. L. Hill, "Seminar on the theory of quantum mechanics" (unpublished), University of Minnesota (1954).

[12] J. Schwinger, "On angular momentum," NYO-3071 (Office of Technical Services, Department of Commerce, Washington, D. C., 1952).

[13] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828–832 (1949).

[14] M. H. Johnson and B. A. Lippmann, Phys. Rev. **77**, 702–705 (1950).

[15] M. H. Johnson and B. A. Lippmann, Phys. Rev. **78**, 329(A) (1950).

[16] W. Furry (private communication).

eter space may be described in a number of ways. For instance, when a rotation is written as the exponential of an antisymmetric matrix, it is natural to parameterize it in terms of the direction cosines of the axes of rotation and the angles of the rotations; the result being a solid ball of radius π with diametrically opposite points upon its surface identified. The Cayley parameterization makes it more natural to use all of 3-space with diametrically opposite points at infinity identified, since the radius of a point is not now ψ, the angle of rotation, but rather $\tan\frac{1}{2}\psi$. This is actually projective 3-space and may be regarded as a gnomonic projection of the hypersphere, upon which diametrically opposite points have been identified. The hypersphere itself becomes the parameter space of the covering group of the three-dimensional rotation group (i.e., the 2×2 unitary unimodular group) if one uses as parameters either the quaternions or the Cayley-Klein parameters.

Turning to the representation theory of the rotation group, we find that there are three ways to derive and discuss the irreducible representations in common use. One is to represent the rotation group as a group of transformations on the spherical harmonics. Another is to allow them to act as a transformation group on the functions of a complex variable through the unitary unimodular group as a group of fractional linear transformations of the complex plane. Finally group theory itself teaches us that the regular representation of the group may be used, acting as a group of substitutions on the arguments of functions defined over the group manifold. Since the group manifold may be taken as the unit hypersphere and since the Haar measure is nothing but the Lebesgue measure induced from 4-space, this means that it may be represented as a group of transformations acting on the hyperspherical harmonics.

Between the left and right regular representations, the left and right translations generate the entire four-dimensional rotation group. The left and right translations do not operate on the hypersphere in the manner in which one might expect, and it is fairly interesting to examine their trajectories. All the left translations commute with all the right translations; and it is only the sums and differences of the infinitesimal operators representing rotations about the coordinate axes (considered as left and right translations on the group manifold) which generate the rotations in the coordinate planes of four-space which one is used to finding in the exponential parameterization of the four-dimensional rotation group. In fact if we use the following table to indicate that, for a given operator, there will be an antisymmetric 4×4 matrix possessing a $+1$ in the position occupied by its name ($-$ indicates a -1),

$$\begin{array}{lll} -L_z & -L_y & A_x \\ & -L_x & A_y \\ & & -A_z \end{array}$$

and if we let $E_i=L_i+A_i$; $H_i=L_i-A_i$, we have the operators generating the left and right translations. The trajectory of any point on the hypersphere, under E_z, for instance, is a great circle of the sphere. In fact, the hypersphere partitions itself into a bundle of great circles under the action of E_z. The H_z trajectories are a family of great circles orthogonal to the E_z trajectories.

It is against the relationship between the three-dimensional rotation group and the hypersphere that one must examine the accidental degeneracy of the two-dimensional isotropic harmonic oscillator. The relationship is just that phase space for the harmonic oscillator is four-dimensional, and if the coordinates are so normalized that the frequency of the oscillator is one rad/sec, the constant energy surfaces are hyperspheres,

$$2H=p_1^2+p_2^2+q_1^2+q_2^2,$$

where $H=$constant. It is easy to calculate the quadratic constants of the motion, which are clearly closed under the Poisson-bracket operation:

$$L=p_1q_1-p_2q_1,$$
$$K=p_1p_2+q_1q_2,$$
$$D=\tfrac{1}{2}(p_1^2+q_1^2)-\tfrac{1}{2}(p_2^2+q_2^2).$$

These satisfy, except for a factor $\frac{1}{2}$, the commutation rules of the three-dimensional rotation group. The significance of L is that it is the angular momentum of the oscillator in the plane, and D is apparently the energy difference between the two oscillators formed by the individual coordinates. The third constant of the motion, K, is a constant which is peculiar to the harmonic oscillator problem, and measures the tendency of one of the coordinates to follow the motion of the other.

Considered as the generators of infinitesimal contact transformations, these constants of the motion have the following significance: L generates infinitesimal rotations of the orbits, while K generates infinitesimal changes in eccentricity while preserving the sum of the squares of the semi-axes. D is a composite of the two, advancing the phase of one oscillator while retarding that of the other. It is while making this interpretation that we find the reason that it is the unitary unimodular group and not the rotation group which is generated by these constants of the motion. If K acts, it takes an orbit—let us say nearly circular—and pinches it down into an orbit of higher and higher eccentricity until it collapses into a straight line. Continued application of K produces again an elliptical orbit, *but now traversed in the opposite sense*, so that it takes a 720° rotation to bring the orbit back into itself. The two-valuedness of the mapping arises from the fact that the orbits are oriented.

The significance of the constants of the motion becomes even more illuninating in quantum mechanics; the operator $K+iL$ is not Hermitian, but is nevertheless a (complex) constant of the motion:

$$K+iL=(p_1-iq_1)(p_2+iq_2).$$

Upon factorization it is revealed as a product of ladder operators belonging to the one-dimensional coordinate oscillators. A quantum of energy is annihilated from one coordinate, but that coordinate is left in an eigenstate. Meanwhile a quantum of energy is created in the other coordinate, which is also left in an eigenstate. All told, the energy of the two-dimensional oscillator has not changed, but nevertheless a quantum of energy has been shifted from one coordinate to the other, which leaves the oscillator in a different eigenstate.

In the rotation group, on the other hand, $K+iL$ is the creation operator for the azimuthal quantum number; it is this relationship which lies at the basis of Schwinger's paper. He uses the harmonic oscillator operator in the form of an occupation number operator for a second-quantized spin $\frac{1}{2}$ particle, but the difference is purely formal. This interpretation of energy transfer between coordinates is also strikingly verified when one writes out the wave functions for the oscillator and applies $K+iL$ to them; then one can watch the nodes disappear from one coordinate and appear in the other.

The idea to interpret $p+iq$ as a complex variable is not new to Hamiltonian mechanics; however when we complexify the phase space in this manner we obtain an interesting result. Namely, the Hamiltonian now acts as multiplication by the complex number e^{it}, and advances the phases of all points uniformly in their orbits. Defining

$$w_i=p_i+iq_i,$$

and forming the ratio w_1/w_2, we find that the time dependence of the coordinates disappears. Geometrically the formation of this ratio is the equivalent of a gnomonic projection, such as the one which carried the quaternion parameterization of the rotation group into the Cayley parameterization. However that was a real gnomonic projection; this one is complex. The Hamiltonian is the analog of H_z, and thus we have a mapping collapsing the orbits of H_z into points; likewise it maps the orbits of the harmonic oscillator into points. This mapping is intimately related to the Euler angle parameterization of the rotation group.

This preliminary gnomonic projection carries complex two-dimensional space (the four real dimensional phase space) into a projective one-dimensional complex space, which we may as well regard as a two-dimensional real space—the Argand diagram of complex variable theory. It in turn is a stereographic projection of the Riemann sphere, of radius $\frac{1}{2}$, so that after dilation we have mapped the hypersphere onto the sphere. This is the celebrated Hopf mapping. Orbits of the harmonic oscillator are collapsed into points, and in no more striking way could the three-dimensional rotation group be exposed as the symmetry group of the harmonic oscillator. The constants K, L, and D go over, under this mapping, into the three angular momentum operators generating infinitesimal rotations about the coordinate axes.

An alternative Hopf mapping is the following: we define

$$z_1 = p_1 + ip_2,$$
$$z_2 = q_1 + iq_2,$$

as two complex variables and carry out the same mapping. Now if we take as new coordinates the quantities

$$Q_1 + iQ_2 = (p_1 + ip_2)/(q_1 + iq_2),$$
$$P_1 + iP_2 = (q_1 + iq_2)^2,$$

the result is a canonical contact transformation and we obtain the coordinates used by Säenz and Laporte. In this coordinate system L, K, and D are also recognizable as angular momentum type operators in virtue of their commutation rules. The presence of the three-dimensional rotation group as a symmetry group in this case is made apparent by the fact that the orbits are now great circles upon the unit sphere, corresponding to the force-free motion of a particle constrained to lie upon the spherical surface. This mapping is particularly interesting for the way it shows how parabolic coordinates enter the problem of the harmonic oscillator and Kepler problem as well. Thus we see why Fock had to work in momentum space in his paper.

The rotation group and the harmonic oscillator are linked in one further fashion. The Hopf mapping first described mapped the trajectories of the harmonic oscillator into points. These trajectories are the same as the trajectories of H_z, if the hyperspherical energy surface and the parameter space of the rotation group are compared.

By this survey, we have seen that the subject of accidental degeneracy is much deeper than it appears at first sight. Although we have only touched upon the two-dimensional isotropic harmonic oscillator at any length, nevertheless there are many other instances of accidental degeneracy whose explanations are very interesting and indicative of further results. For instance, the degeneracy of the various types of rotors is not at all accidental—one has to realize that rotations are parameterized by rotations, and that the hypersphere is the proper configuration space for the problem, and not three-dimensional space. Thus, the symmetry group of the spherical rotor is the four-dimensional rotation group. The E operators correspond to rotation of the space-fixed axes and the H operators to rotation with respect to body-fixed axes. The motion of a spherical body is clearly insensitive to the way we set up a reference coordinate system in space, as well as the orientation with which we attach a coordinate system to the body to mark its motion. Since the symmetric rotor has but cylindrical symmetry only H_z remains as a symmetry operator, rotations about the other two body-fixed axes being lost, although a reflective symmetry remains. Thus the symmetry group of the symmetric rotor is generated by the E's and H_z and hence is $O_3 \times O_2$ rather than O_2 alone. Even the asymmetric rotor retains the four-group (the axes may still be reversed if not rotated by arbitrary amounts) and the space-fixed rotations as a symmetry group.