Discrete

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$\langle n|m\rangle = 1\,\delta_{nm}$$

Continuous

$$\mathbb{1} = \int_{All} dx\, |x\rangle\langle x|$$

$$= \int_{All} dp\, |p\rangle\langle p|$$

$$|\Psi\rangle \equiv \underbrace{\int_{All} dx\, |x\rangle}_{\mathbb{1}} \langle x|\Psi\rangle = \int_{All} dx\, \Psi(x)\, |x\rangle$$

$$\langle q|p\rangle \equiv \delta(q-p) = \langle q|\mathbb{1}|p\rangle = \int_{All} dx\, \langle q|x\rangle\langle x|p\rangle$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx\, e^{\frac{i(p-q)x}{\hbar}} = \delta(p-q)$$

$$\langle x|\hat{X}|\Psi\rangle = \int_{All} dy\, \langle x|\underbrace{\hat{X}|y\rangle}_{y|y\rangle}\langle y|\Psi\rangle$$

$$= \int_{All} dy\, y\, \underbrace{\langle x|y\rangle}_{\delta(x-y)}\Psi(y) = x\,\Psi(x)$$

$$\langle x|\hat{p}\,\hat{\mathbb{1}}|\psi\rangle = \int_{All} dp \, \langle x|\boxed{\hat{p}\,|p\rangle}\langle p|\psi\rangle$$

$$\hat{\mathbb{1}}$$

$$= \int_{All} dp \; p \, \langle x|p\rangle \langle p|\psi\rangle$$

$$= \int_{All} dp \; p \underbrace{\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}}_{} \langle p|\psi\rangle$$

$$= \frac{\hbar}{i}\frac{d}{dx}\left[\frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}\right] = \frac{\hbar}{i}\frac{d}{dx}\langle x|p\rangle$$

$$= \int_{All} dp \, \frac{\hbar}{i}\frac{d}{dx} \langle x|p\rangle\langle p|\psi\rangle = \frac{\hbar}{i}\frac{d}{dx}\langle x|\psi\rangle$$

$$\mathbb{1}$$

$$= \frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

$\hat{p}$ in $|x\rangle$ basis is $\frac{\hbar}{i}\frac{d}{dx}$

Show that $\hat{x}$ in $|p\rangle$ basis is $i\hbar\frac{d}{dp}$

i.e. $\langle p|\hat{x}|\psi\rangle = i\hbar\frac{d}{dp}\tilde{\psi}(p)$.

Canonical Commutation Relation

$$[\hat{x}, \hat{p}] \equiv \hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar\hat{\mathbb{1}}$$

derive in $|x\rangle$ basis

$$[\hat{x}, \hat{p}]\psi(x) = \left[x\left(\frac{\hbar}{i}\frac{d}{dx}\right) - \frac{\hbar}{i}\frac{d}{dx}x\right]\psi(x)$$

$$= x\frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}\underbrace{\frac{d}{dx}[x\,\psi(x)]}_{1\psi(x) + x\frac{d\psi}{dx}}$$

$$= \frac{\hbar}{i}\left[\cancel{x\psi'} - \psi - \cancel{x\psi'}\right] = -\frac{\hbar}{i}\psi(x)$$

$$= i\hbar\,\psi(x) \quad \forall\,\psi(x) \implies [\hat{x}, \hat{p}] = i\hbar\hat{\mathbb{1}}$$

Eigenvalues of hermitian operators are real. $\hat{H} = \hat{H}^\dagger$

① $\hat{H}|n\rangle = n|n\rangle$ eigenvalue n, eigenket $|n\rangle$

hermitian conjugate of both sides

$$(\hat{H}|n\rangle)^\dagger = \langle n|\hat{H}^\dagger = \langle n|\hat{H} = n^*\langle n|$$

② $\langle n|\hat{H} = \langle n|\, n^*$

Now $\langle n|$ with ①, ② with $|n\rangle$

$$\langle n|\hat{H}|n\rangle = n\langle n|n\rangle = n^*\langle n|n\rangle$$

$\Rightarrow n = n^*$; $n \in$ Reals

Eigenkets of hermitian ops with different eigenvalues are orthogonal.

$n \neq b$

① $\hat{H}|n\rangle = n|n\rangle$

$\hat{H}|b\rangle = b|b\rangle$ ← dagger

$\langle b|\hat{H}^{+} = \langle b| b^{*}$

② $\langle b|\hat{H} = \langle b| b$

$\langle b|$ with ①, ② with $|n\rangle$

$\langle b|\hat{H}|n\rangle = n\langle b|n\rangle = b\langle b|n\rangle$ subtract

$0 = \underbrace{(n-b)}_{\neq 0}\underbrace{\langle b|n\rangle}_{\text{zero}} \Longrightarrow \langle b|n\rangle = 0$ orthogonal

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Degenerate Eigenvalues? $\hat{A}|\chi\rangle = a|\chi\rangle$
$\hat{A}|\varphi\rangle = a|\varphi\rangle$

can always choose perpendicular kets in the orthogonal eigensubspace: span$(|\chi\rangle, |\varphi\rangle)$

1-dim non degeneracy theorem

Bound states in one-dimensional potentials are not degenerate if $\forall x > x_0$, $V(x) - E \geq M^2$
(V bounded from below)

Assume: $\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_1(x)}{dx^2} + V(x)\psi_1(x) = E\psi_1(x)\right] * \psi_2(x)$

$\left[-\frac{\hbar^2}{2m}\frac{d^2\psi_2(x)}{dx^2} + V(x)\psi_2(x) = E\psi_2(x)\right] * \psi_1(x)$ subtract

$$\psi_1\psi_2'' - \psi_2\psi_1'' = 0$$

integrate

$$\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x) = \text{constant}$$

derivative $\cancel{\psi_1'\psi_2'} + \psi_1\psi_2'' - \cancel{\psi_2'\psi_1'} - \psi_2\psi_1'' = 0$

For many potentials (non-pathological), if ψ_1 and ψ_2 vanish at the same point (like $x \to \infty$), then constant $= 0$.

Then $\psi_1(x)\psi_2'(x) = \psi_2(x)\psi_1'(x)$

$$\psi_1(x)\frac{d\psi_2}{dx} = \psi_2(x)\frac{d\psi_1}{dx}$$

$$\Rightarrow \int\frac{d\psi_1}{\psi_1} = \int\frac{d\psi_2}{\psi_2}$$

$$\Rightarrow \ln\psi_1 = \ln\psi_2 + \text{constant}$$

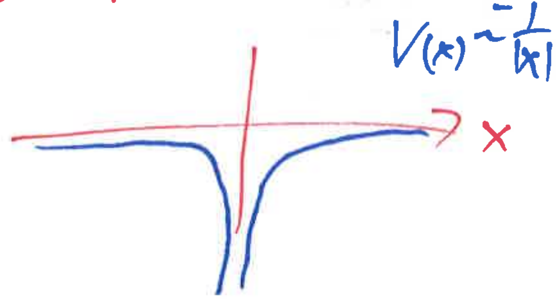
exp both sides

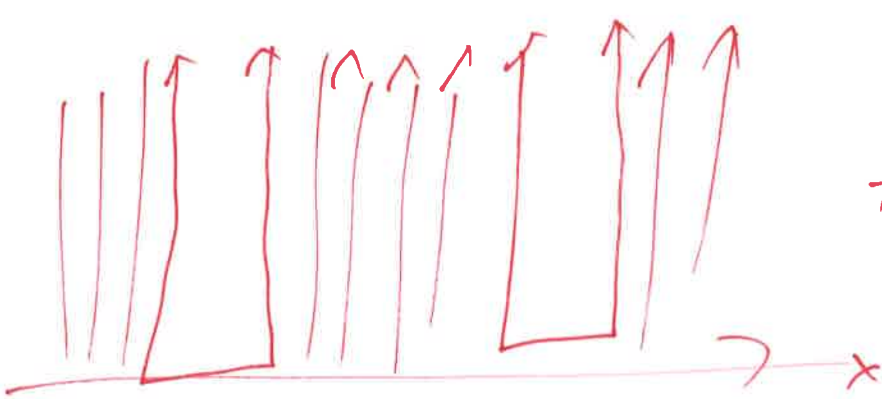
$$\psi_1(x) = \psi_2(x)e^{b} = c\,\psi_2(x)$$

See Messiah pp. 98–106

Exceptions: $V(x)$ not bounded from below

one-dimensional hydrogen





two disconnected infinite square wells.