Common Potentials + Wavefunction Solutions $\psi(x)$ to the Schrödinger Eq.

- **TISE** $$\underbrace{\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x)}_{H(x)\psi(x)} = E\psi(x)$$

2nd order, linear in $\psi(x)$, homogeneous, Ordinary D.E.
↳ two linearly independent solutions

- **TDSE** $$\frac{-\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,\cancel{t})\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

could be constant.
⇒ change $\psi(x)$ by phase.

① Free Particle $V(x)=0$ everywhere

$$\psi(x) = A e^{\frac{ipx}{\hbar}} + B e^{-\frac{ipx}{\hbar}}$$

↑ "right moving" ↑ "left moving"

plane waves
not in L^2
not square integrable

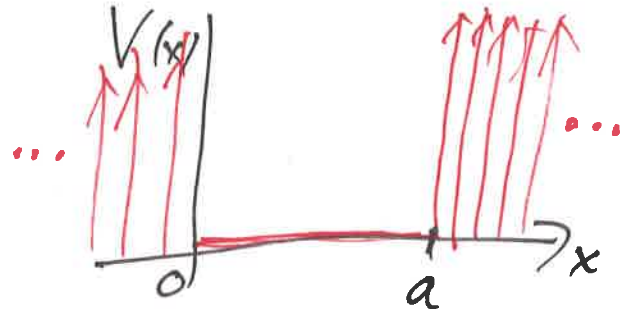
Can't normalize $\int_{All} dx\, |\psi(x)|^2 = 1$, delta function or continuum normalization

$$E = \frac{p^2}{2m} + 0 \Rightarrow p = \pm\sqrt{2mE} \quad E > 0$$

Non-denumerably many solutions, all scattering states.

3-dim $\vec{p}\cdot\vec{r}$, $\vec{k} = \frac{\vec{p}}{\hbar}$, $\omega = \frac{E}{\hbar}$, $\lambda = \frac{2\pi}{|\vec{k}|}$ DeBroglie wavelength

② Infinite Square Well (1-dim)
Particle in a box



$$V(x) = \begin{cases} 0 \text{ (or constant)}, & 0 < x < a \\ \infty, & \text{elsewhere} \end{cases}$$

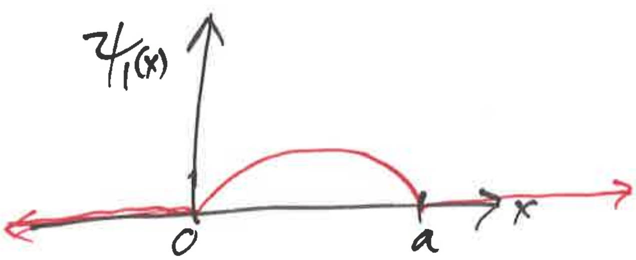
Only bound states, no scattering state, Energies are quantized E_n

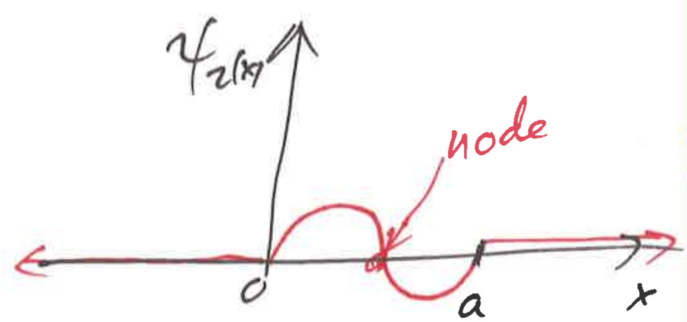
↑ Denumerably infinitely many

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right), & 0 < x < a \\ 0, & \text{elsewhere} \end{cases} \qquad n = 1, 2, 3, \cdots \text{ not } 0$$

$$\langle \psi_n | \psi_m \rangle = \delta_{nm}, \quad \underbrace{|\psi_n|}_{\text{norm}} = 1$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, 3, \cdots$$





$\psi_n(x)$ continuous, $\psi_n'(x)$ discontinuous

2+3 dim Infinite Square Well

$$V(x) = \begin{cases} 0 \ \text{(or constant)}, & 0<x<a,\ 0<y<b,\ 0<z<c \\ \infty, & \text{elsewhere} \end{cases}$$

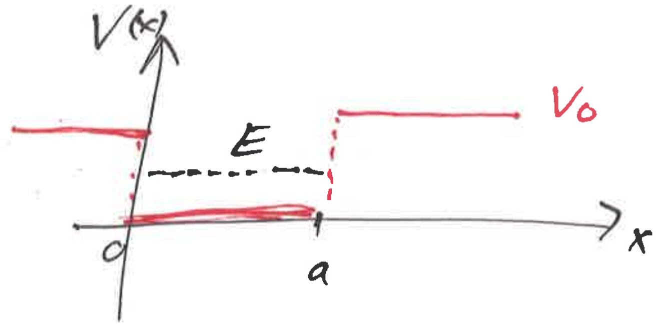
$$\psi_{npq}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{p\pi y}{b}\right)\sin\left(\frac{q\pi z}{c}\right)$$

$$E_{npq} = \frac{\pi^2\hbar^2}{2m}\left(\frac{n^2}{a^2}+\frac{p^2}{b^2}+\frac{q^2}{c^2}\right), \quad n,p,q = 1,2,3,\ldots \quad \text{none} = 0$$

---

③ 1-dim Finite Square Well

$$V(x) = \begin{cases} 0 \ (\leftarrow \text{add } c), & 0<x<a \\ V_0 \ (\leftarrow \text{add } c), & \text{elsewhere} \end{cases}$$



$\psi(x)$ continuous, $\psi'(x)$ continuous

Always at least one bound state (symmetric) no matter how wide or deep.

Finitely many bound states, $E<V_0$

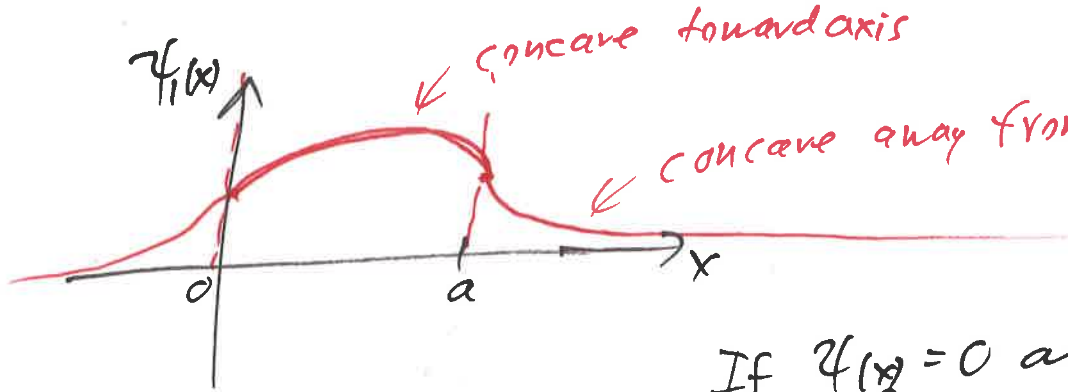
Non-denumerably many scattering states, $E>V_0$

↑ [redacted]

If $E < V_0$, then

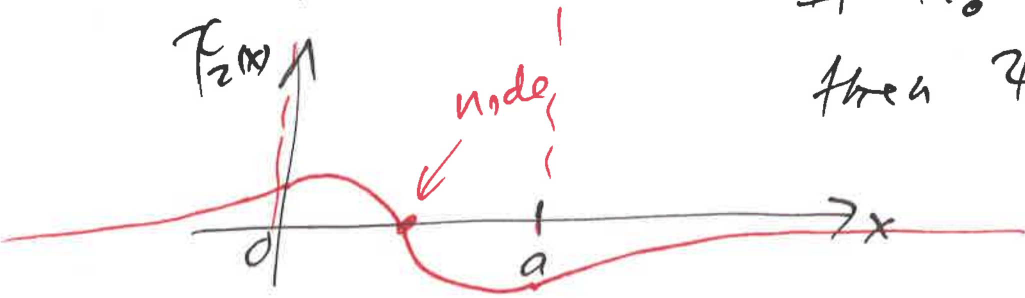
classically allowed region : $0 < x < a$ — oscillatory sin, cos

" forbidden " : elsewhere — real exponentials



If $\psi(x_0) = 0$ and $\psi'(x_0) = 0$

then $\psi(x) = 0$ everywhere



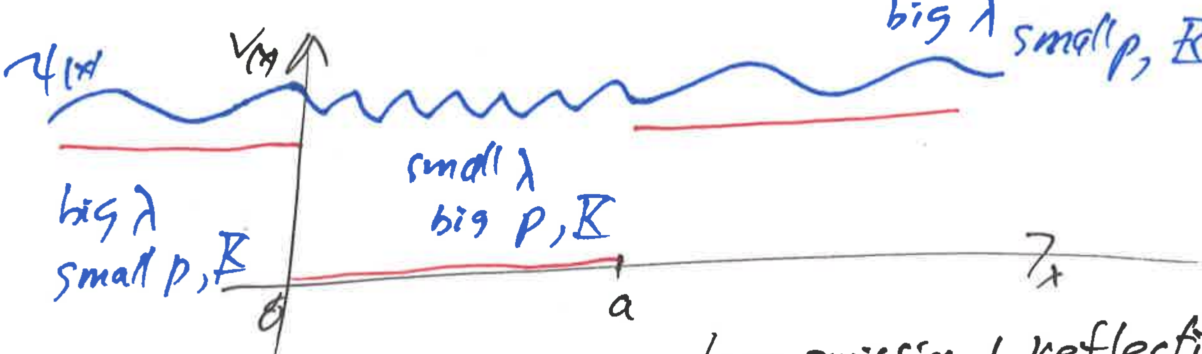
$\psi_n(x)$, E_n there no closed form solutions → numerically

because continuity of ψ, ψ' result in

transcendental Equations.

Scattering states $E > V_0$

$$p = \frac{h}{\lambda}$$

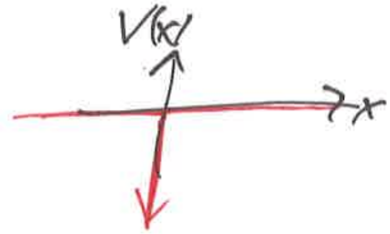


$$K = \frac{p^2}{2m}$$

transmission + reflection coefficients

$$T + R = 1$$

9 Delta function well

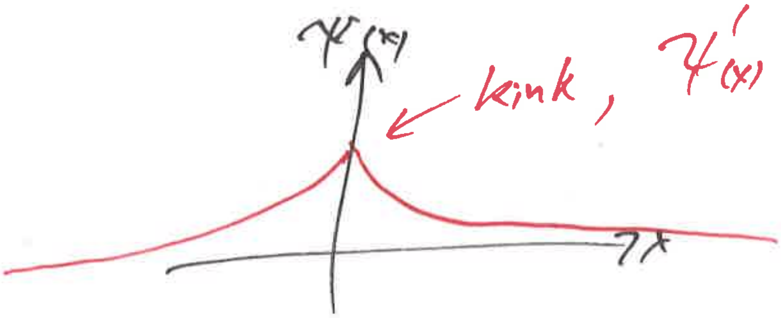


$$V(x) = -\lambda\,\delta(x)$$

$E>0$, nondenumerably many scattering states, any real E

$E<0$, exactly one bound state.

$$\psi_b(x) = \sqrt{\frac{m\lambda}{\hbar^2}}\, e^{-\frac{m\lambda}{\hbar^2}|x|}, \quad E = -\frac{m\lambda^2}{2\hbar^2} < 0$$



kink, $\psi'(x)$ is discontinuous at $x=0$