# Schwarz Inequality

Consider two kets $|\varphi_1\rangle$ and $|\varphi_2\rangle$
not necessarily orthogonal, nor normalized, but
neither is the zero ket : $\langle \text{zero}|\text{zero}\rangle = 0$

Define $|\psi\rangle = |\varphi_1\rangle + \lambda|\varphi_2\rangle$

$$\langle\psi|\psi\rangle = \langle\varphi_1|\varphi_1\rangle + \lambda\langle\varphi_1|\varphi_2\rangle + \lambda^*\langle\varphi_2|\varphi_1\rangle + \lambda\lambda^*\langle\varphi_2|\varphi_2\rangle \geq 0$$

choose $\lambda = -\dfrac{\langle\varphi_2|\varphi_1\rangle}{\langle\varphi_2|\varphi_2\rangle}$ , $\lambda^* = -\dfrac{\langle\varphi_1|\varphi_2\rangle}{\langle\varphi_2|\varphi_2\rangle}$

$$\langle\psi|\psi\rangle = \langle\varphi_1|\varphi_1\rangle - \frac{\langle\varphi_2|\varphi_1\rangle\langle\varphi_1|\varphi_2\rangle}{\langle\varphi_2|\varphi_2\rangle} - \cancel{\frac{\langle\varphi_1|\varphi_2\rangle\langle\varphi_2|\varphi_1\rangle}{\langle\varphi_2|\varphi_2\rangle}} + \cancel{\frac{\langle\varphi_2|\varphi_1\rangle\langle\varphi_1|\varphi_2\rangle}{\langle\varphi_2|\varphi_2\rangle}} \geq 0$$

$$\langle\psi|\psi\rangle = \langle\varphi_1|\varphi_1\rangle - \frac{\langle\varphi_1|\varphi_2\rangle\langle\varphi_2|\varphi_1\rangle}{\langle\varphi_2|\varphi_2\rangle} \geq 0$$

multiply $\langle\varphi_2|\varphi_2\rangle$

$$\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle - \langle\varphi_1|\varphi_2\rangle\langle\varphi_2|\varphi_1\rangle \geq 0$$

$$\langle\varphi_1|\varphi_1\rangle\langle\varphi_2|\varphi_2\rangle \geq |\langle\varphi_1|\varphi_2\rangle|^2$$

equality iffi $|\varphi_1\rangle = -\lambda|\varphi_2\rangle$

# Generalized Uncertainty Principle

for observable $\hat{A}$ (hermitian $\hat{A}^\dagger = \hat{A}$)

in state $|\psi\rangle$

$\langle\psi|\hat{A}|\psi\rangle$

$\langle\hat{A}\rangle$ is a number

$$\sigma_A^2 = (\Delta A)^2 = \langle(\hat{A}-\langle\hat{A}\rangle)^2\rangle = \langle\psi|(\hat{A}-\langle\hat{A}\rangle)^2|\psi\rangle$$

$$= \langle(\hat{A}-\langle\hat{A}\rangle)\psi|(\hat{A}-\langle\hat{A}\rangle)\psi\rangle = \langle f|f\rangle$$

where $f \equiv (\hat{A}-\langle A\rangle\mathbb{1})|\psi\rangle$

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$$\sigma_B^2 = \langle(\hat{B}-\langle\hat{B}\rangle)^2\rangle \equiv \langle g|g\rangle$$

where $g \equiv (\hat{B}-\langle\hat{B}\rangle\mathbb{1})|\psi\rangle$

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Schwarz Inequality

Assume $\langle\psi|\psi\rangle = 1$ normalized

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

For any complex number $z$: $|z|^2 = [Re(z)]^2 + [Im(z)]^2$
$\geq [Im(z)]^2$

Let $z \equiv \langle f|g\rangle$

$|\langle f|g\rangle|^2$

$$Im(z) = \frac{z-z^*}{2i}, \quad Re(z) = \frac{z+z^*}{2}$$

$$|z|^2 \geq [\mathrm{Im}(z)]^2$$

$$\sigma_A^2\sigma_B^2 \geq |\langle f|g\rangle|^2 \geq \left(\frac{\langle f|g\rangle - \langle g|f\rangle}{2i}\right)^2$$

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$$\langle f|g\rangle = \langle(\hat{A}-\langle\hat{A}\rangle)\psi|(\hat{B}-\langle\hat{B}\rangle)\psi\rangle$$

$$= \langle\psi|(\hat{A}-\langle\hat{A}\rangle)(\hat{B}-\langle\hat{B}\rangle)|\psi\rangle$$

$$= \langle\psi|(\underset{\uparrow\,op}{\hat{A}}\underset{\uparrow\,op}{\hat{B}} - \underset{\uparrow\,op}{\hat{A}}\underset{\uparrow\,\#}{\langle\hat{B}\rangle} - \underset{\uparrow\,op}{\hat{B}}\underset{\uparrow\,\#}{\langle\hat{A}\rangle} + \underset{\uparrow\,\#}{\langle\hat{A}\rangle}\underset{\uparrow\,\#}{\langle\hat{B}\rangle})|\psi\rangle$$

$$= \langle\psi|\hat{A}\hat{B}|\psi\rangle - \langle\hat{B}\rangle\underbrace{\langle\psi|\hat{A}|\psi\rangle}_{\langle\hat{A}\rangle} - \langle\hat{A}\rangle\langle\psi|\hat{B}|\psi\rangle \leftarrow \langle\hat{B}\rangle$$
$$+ \langle\hat{A}\rangle\langle\hat{B}\rangle\underbrace{\langle\psi|\psi\rangle}_{1}$$

$$= \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\rangle\langle\hat{A}\rangle - \cancel{\langle\hat{A}\rangle\langle\hat{B}\rangle} + \cancel{\langle\hat{A}\rangle\langle\hat{B}\rangle}$$

$$= \langle f|g\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

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$$\langle g|f\rangle = \langle\hat{B}\hat{A}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle$$

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$$\langle f|g\rangle - \langle g|f\rangle = \langle\hat{A}\hat{B}\rangle - \langle\hat{B}\hat{A}\rangle = \langle[\hat{A},\hat{B}]\rangle$$

$$\sigma_A^2\sigma_B^2 \geq \left(\frac{\langle[\hat{A},\hat{B}]\rangle}{2i}\right)^2$$

Let $\hat{A} = \hat{x}$ , $\hat{B} = \hat{p}$

$$\sigma_x^2 \sigma_p^2 \geq \left( \frac{\langle [\hat{x}, \hat{p}] \rangle_\psi}{2i} \right)^2 = \left( \frac{i\hbar}{2i} \right)^2 = \left( \frac{\hbar}{2} \right)^2$$

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \quad \checkmark$$

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# Schrödinger vs. Heisenberg Pictures

$$|\psi_s(t)\rangle = \hat{U}(t, t_0) |\psi_s(t_0)\rangle$$

↖ unitary time evolution operator

$$\hat{U}(t, t_0) = e^{\frac{-i \hat{H}_s (t - t_0)}{\hbar}}$$

$\hat{A}_s(t)$ operator in S. picture

$$\langle \psi_s(t) | \hat{A}_s(t) | \psi_s(t) \rangle$$

$$= \langle \psi_s(t_0) | \underbrace{\hat{U}^\dagger(t, t_0) | \hat{A}_s(t) | \hat{U}(t, t_0)}_{\equiv \hat{A}_H(t)} | \psi_s(t_0) \rangle$$

# Three dimensions

TDSE: $-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r},t)\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)$

2nd-order, linear in $\Psi$, homogeneous, Partial D.E

Laplacian $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in Cartesian coords

$[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$ e.g. $[\hat{y}, \hat{p}_x] = 0$ // $[\hat{r}_i, \hat{r}_j] = 0 = [\hat{p}_i, \hat{p}_j]$

$\Rightarrow \sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}$, $\sigma_y \sigma_{p_y} \geq \frac{\hbar}{2}$, but $\sigma_x \sigma_{p_y}$ is unrestricted

We saw previously the 3-dim infinite square well, 3-dim quantum harmonic oscillator, both in Cartesian