If $|\psi\rangle$ is an eigen ~~function~~ ket of $\hat{L}^2$ with eigenvalue λ,
$\hat{L}^2|\psi\rangle = \lambda|\psi\rangle$

then so is $|\varphi\rangle \equiv \hat{L}_\pm|\psi\rangle$

$$\hat{L}^2|\varphi\rangle = \hat{L}^2\hat{L}_\pm|\psi\rangle = \hat{L}_\pm \underbrace{\hat{L}^2|\psi\rangle}_{\lambda|\psi\rangle} = \lambda\hat{L}_\pm|\psi\rangle = \lambda|\varphi\rangle$$

they commute

If $|\psi\rangle$ is an eigenket of $\hat{L}_z$ with eigenvalue μ,
$\hat{L}_z|\psi\rangle = \mu|\psi\rangle$

then $|\varphi\rangle = \hat{L}_\pm|\psi\rangle$ is also an eigenket of $\hat{L}_z$, but with a different eigenvalue.

$$\hat{L}_z|\varphi\rangle = \hat{L}_z\hat{L}_\pm|\psi\rangle = \underbrace{(\hat{L}_z\hat{L}_\pm - \hat{L}_\pm\hat{L}_z)}_{[\hat{L}_z, \hat{L}_\pm] = \pm\hbar\hat{L}_\pm}|\psi\rangle + \hat{L}_\pm\underbrace{\hat{L}_z|\psi\rangle}_{\mu|\psi\rangle}$$

$$= \pm\hbar\hat{L}_\pm|\psi\rangle + \hat{L}_\pm\mu|\psi\rangle = (\mu\pm\hbar)\hat{L}_\pm|\psi\rangle$$

$$= (\mu\pm\hbar)|\varphi\rangle$$

Can not generate an infinite number of states —
there is a "top rung" and "bottom rung" to the ladder.

$$\underbrace{\langle\psi|\hat{L}^2|\psi\rangle}_{\lambda\langle\psi|\psi\rangle} = \langle\hat{L}^2\rangle = \langle\hat{L}_x^2\rangle + \langle\hat{L}_y^2\rangle + \langle\hat{L}_z^2\rangle \leftarrow \mu^2\underbrace{\langle\psi|\psi\rangle}_{1}$$

$$\langle\psi\hat{L}_x|\hat{L}_x\psi\rangle = \langle(\hat{L}_x\psi)|(\hat{L}_x\psi)\rangle = \||\hat{L}_x\psi\rangle\|^2 \geq 0$$

$$\langle\psi\hat{L}_y|\hat{L}_y\psi\rangle \geq 0$$

$$\lambda = \langle\hat{L}_x^2\rangle + \langle\hat{L}_y^2\rangle + \mu^2 \geq \mu^2$$

Call the top rung state $|\chi\rangle$. then $\hat{L}_+|\chi\rangle = \emptyset$

$\hat{L}^2|\chi\rangle = \lambda|\chi\rangle$ $\qquad$ $\hat{L}_z|\chi\rangle = \hbar\ell\,|\chi\rangle$

↑ maximum $\hat{L}_z$ eigenvalue

$$\hat{L}_\pm \hat{L}_\mp = (\hat{L}_1 \pm i\hat{L}_2)(\hat{L}_1 \mp i\hat{L}_2) =$$

$$= \hat{L}_1^2 + \hat{L}_2^2 \mp i\underbrace{(\hat{L}_1\hat{L}_2 - \hat{L}_2\hat{L}_1)}_{[\hat{L}_1,\hat{L}_2] = i\hbar\hat{L}_3} = \hat{L}_1^2 + \hat{L}_2^2 \mp i(i\hbar)\hat{L}_3$$

$$= \hat{L}^2 - \hat{L}_3^2 \pm \hbar\hat{L}_3$$

$$\Longrightarrow \hat{L}^2 = \hat{L}_\pm\hat{L}_\mp + \hat{L}_3^2 \mp \hbar\hat{L}_3$$

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lower signs: $\hat{L}^2|\chi\rangle = (\hat{L}_-\hat{L}_+ + \hat{L}_3^2 + \hbar\hat{L}_3)|\chi\rangle$

$$\begin{aligned} \lambda|\chi\rangle \;=\; & (\emptyset + \hbar^2\ell^2 + \hbar^2\ell)|\chi\rangle \\ = \; & \hbar^2\ell(\ell+1)\,|\chi\rangle \end{aligned}$$

$$\boxed{\lambda = \hbar^2\ell(\ell+1)}$$

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upper signs: Call the bottom rung state $|\beta\rangle$, then $\hat{L}_-|\beta\rangle = \emptyset$

Say $\hat{L}_z|\beta\rangle = \hbar\bar{\ell}\,|\beta\rangle$ $\qquad$ minimum eigenvalue of $\hat{L}_z$

$$\begin{aligned} \hat{L}^2|\beta\rangle &= (\hat{L}_+\hat{L}_- + \hat{L}_3^2 - \hbar\hat{L}_3)|\beta\rangle \\ \lambda|\beta\rangle &= (\emptyset + \hbar^2\bar{\ell}^2 - \hbar^2\bar{\ell})|\beta\rangle \end{aligned}$$

$$\boxed{\lambda = \hbar^2\bar{\ell}(\bar{\ell}-1)}$$

$\Rightarrow$ Either $\bar{l} = l+1$ $\leftarrow$ but this higher than the top rung.

or $\boxed{\bar{l} = -l}$ ✓

Eigenvalues of $\hat{L}_3$ are $m\hbar$ where m goes from $-l$ to $+l$ in integer steps. l need not be integral.

The number of steps is integral. $\Rightarrow l = -l + N$

$l = \frac{N}{2}$ $\Rightarrow$ l must be integral or half integral.

Label states by $l, m \Rightarrow |l, m\rangle$

where $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \cdots$

and $m = \underbrace{-l, -l+1, \cdots l-1, l}_{2l+1 \text{ values of } m \text{ \# rungs on ladder.}}$

$$\hat{L}^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle, \quad \hat{L}_z |l, m\rangle = \hbar m |l, m\rangle$$

$|l, m\rangle$ if l is integral then the coordinate representation is $Y_{lm}(\theta, \varphi)$ spherical harmonics

If $|\ell, m\rangle$ is normalized: $\langle \ell, m | \ell, m\rangle = 1$ then

$\hat{L}_+ |\ell m\rangle$ is <u>not</u> normalized

$$\left\| \hat{L}_+ |\ell, m\rangle \right\|^2 = \langle (\hat{L}_+ |\ell m\rangle) | \hat{L}_+ |\ell m\rangle$$

$$= \langle \ell m | \hat{L}_+^\dagger \hat{L}_+ |\ell m\rangle = \langle \ell m | \hat{L}_- \hat{L}_+ |\ell m\rangle$$

$$= \langle \ell m | (\hat{L}^2 - \hat{L}_3^2 - \hbar \hat{L}_3) |\ell m\rangle$$

$$(\hbar^2 \ell(\ell+1) \quad -\hbar^2 m^2 \quad -\hbar^2 m) \underbrace{\langle \ell m | \ell m\rangle}_{1}$$

$$\hat{L}_+ |\ell m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m+1)}\, |\ell, m+1\rangle$$

$$\hat{L}_- |\ell m\rangle = \hbar \sqrt{\ell(\ell+1) - m(m-1)}\, |\ell, m-1\rangle$$

Robertson-Schrödinger Inequality

$$\sigma_{L_x} \sigma_{L_y} \geq \frac{\hbar}{2} \left| \langle \hat{L}_z \rangle \right|$$

$$\equiv \Delta_{L_x} \Delta_{L_y}$$

If $V(\vec{r}) = V(r)$ central potential then

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{R})$$ commutes with both $\hat{L}^2$ and $\hat{\vec{L}}$.

$\{\hat{H}, \hat{L}^2, \hat{L}_z\}$ are compatible observables.

z say

$$[\hat{H}, \vec{L}] = 0 \Longrightarrow \begin{cases} \hat{\vec{L}} \text{ is conserved } \frac{d\langle \vec{L}\rangle}{dt} = 0 \\ \hat{H} \text{ is a scalar operator} \end{cases}$$

Coordinate Representation

Cartesian: $\hat{L}_x = \hat{y}\hat{P}_z - \hat{z}\hat{P}_y = y\frac{\hbar}{i}\frac{\partial}{\partial z} - z\frac{\hbar}{i}\frac{\partial}{\partial y}$

$$= \frac{\hbar}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

Spherical Polar coordinates $\vec{r} = r\hat{r}$

$$\hat{\vec{L}} = \hat{\vec{R}} \times \frac{\hbar}{i}\vec{\nabla} = \frac{\hbar}{i}\vec{r} \times \vec{\nabla}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\hat{\vec{L}} = \frac{\hbar}{i}\Big[r\,\cancelto{0}{(\hat{r}\times\hat{r})}\frac{\partial}{\partial r} + \underbrace{(\hat{r}\times\hat{\theta})}_{\hat{\varphi}}\frac{\partial}{\partial\theta} + \underbrace{(\hat{r}\times\hat{\varphi})}_{-\hat{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\hat{\vec{L}} = -i\hbar\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$\hat{\theta} = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

$$\Rightarrow \hat{L}_3 = \hat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}$$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$\hat{L}_+ = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$

$$\hat{L}_- = \hat{L}_+^\dagger = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$$ looks wrong but correct.

(Remember $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ is hermitian)

$|\ell, m\rangle$ other bases

$\underline{\ell = 0}$ $\quad |00\rangle$, $\quad \hat{L}_3|00\rangle = 0$, $\hat{\vec{L}}|00\rangle = 0$

$\hat{L}_2|00\rangle = 0$.

all operators ~~a~~ are 1×1 matrices, all 0

The Lie algebra $so(3)$ is rank 1.

$\Rightarrow$ 1 Casimir operator $\Rightarrow \hat{L}^2$