$$\vec{\mu}_e = -g\,\mu_B \frac{\vec{S}}{\hbar} \qquad \mu_B \equiv \left(\frac{e\hbar}{2m_e}\right) = 5.788 \times 10^{-5} \frac{eV}{T}$$

Bohr magneton

g is the g-factor — Dirac gives exactly 2

QFT (Dirac + Loop corrections, quantize the Electromagnetic field)

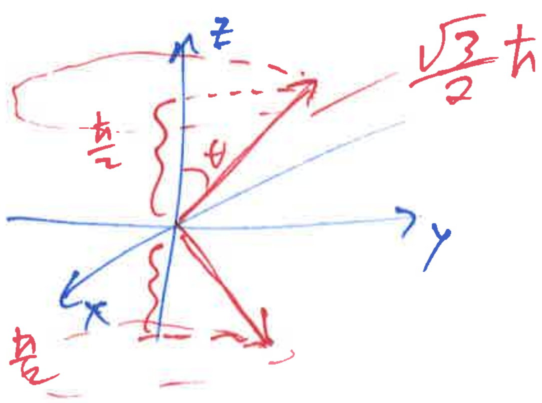
$g_{theory} = 2.002\ 319\ 304\ 363\ 286\ (1528)$ T. Kinoshita α^5 order

for e^-

In a magnetic field $B\hat{z}$

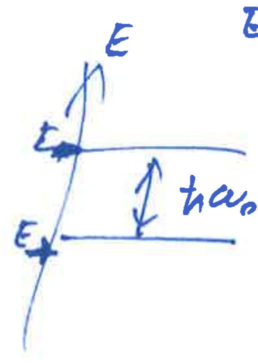
$\vec{\mu}$ "up" along $\hat{z}$: $|+\rangle_z$ — low energy $\hat{H}|+\rangle_z = -\mu B|+\rangle_z$ (E_+)

$\vec{\mu}$ "down" along $\hat{z}$: $|-\rangle_z$ — high energy $\hat{H}|-\rangle_z = +\mu B|-\rangle_z$ (E_-)



$E_+ = -\mu B = -\hbar\frac{\omega_0}{2}$

$E_- = +\mu B = +\frac{\hbar\omega_0}{2}$



At time $t=0$: $|\psi_0\rangle = \alpha|+\rangle_z + \beta|-\rangle_z$

$$= \cos\left(\frac{\theta}{2}\right)e^{-i\frac{\varphi}{2}}|+\rangle_z + \sin\left(\frac{\theta}{2}\right)e^{+i\frac{\varphi}{2}}|-\rangle_z$$

$$|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right)e^{-i\frac{\varphi}{2}}e^{-\frac{i\omega_0 t}{2}}|+\rangle_z + \sin\left(\frac{\theta}{2}\right)e^{i\frac{\varphi}{2}}e^{+\frac{i\omega_0 t}{2}}|-\rangle_z$$

$$\langle\psi(t)|\hat{S}_z|\psi(t)\rangle = \frac{\hbar}{2}\cos\theta \quad \text{no time dependence}$$

$$\left.\begin{aligned}\langle\psi(t)|\hat{S}_x|\psi(t)\rangle &= \frac{\hbar}{2}\sin\theta\cos(\varphi+\omega_0 t)\\ \langle\psi(t)|\hat{S}_y|\psi(t)\rangle &= \frac{\hbar}{2}\sin\theta\sin(\varphi+\omega_0 t)\end{aligned}\right\} \text{Larmor precession.}$$

Adjoint Representation of $\hat{L}$

$$(\hat{L}_i)_{jk} = -i\hbar\,\epsilon_{ijk} \quad \Big| \quad \hat{L}_x = \hat{L}_1 = \hbar\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}$$

$$\hat{L}_y = \hat{L}_2 = \hbar\begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \hat{L}_z = \hat{L}_3 = \hbar\begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar\sum_k \epsilon_{ijk}\hat{L}_k \qquad S^{-1}L_{z\,\text{diag}}\,S = L_{Adj}$$

Pauli Matrices

$\hat{\vec{S}} = \frac{\hbar}{2}\underbrace{\hat{\vec{\sigma}}}$ 3×1 column vector of 2×2 matrices

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hermitian $\sigma_i^\dagger = \sigma_i$

determinant $\det(\sigma_i) = -1$, trace $= \mathrm{tr}(\sigma_i) = 0$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{I}_{2\times 2}$$

$\sigma_x \sigma_y = i\,\sigma_z$ (+ cyclic permutation $x \to y \to z \to x$)

Anticommutator $\{\sigma_x, \sigma_y\} \equiv \sigma_x\sigma_y + \sigma_y\sigma_x = 0$

different Pauli matrices anticommute

$\{A, B\} = [A, B]_+$

$$\sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_j \sigma_k = i \sum_{\ell=1}^{3} \epsilon_{jk\ell}\, \sigma_\ell + \delta_{jk}\, \mathbb{I}$$

Anticommutator $[\sigma_x, \sigma_y] \equiv \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\,\sigma_z$

+ cyclic permutations

$$\left.\begin{aligned} [\sigma_j, \sigma_k] &= 2i \sum_{\ell=1}^{3} \epsilon_{jk\ell}\, \sigma_\ell \\ \{\sigma_j, \sigma_k\} &= 2\,\delta_{jk}\, \mathbb{I} \end{aligned}\right\}$$

— su(2) Lie Algebra

$j, k = 1, 2, 3$ independently

$$\mathrm{tr}(\sigma_i) = \emptyset$$

$$\mathrm{tr}(\sigma_\ell \sigma_j) = 2\delta_{\ell j}$$

$$\mathrm{tr}(\sigma_\ell \sigma_j \sigma_k) = 2i\,\epsilon_{\ell jk}$$

$$\mathrm{tr}(\sigma_i \sigma_j \sigma_k \sigma_\ell) = 2(\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = (\vec{A}\cdot\vec{B})\mathbb{I} + i\,\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

proof

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \sum_{j,k=1}^{3} \sigma_j A_j \sigma_k B_k =$$

$$= \sum_{jk} A_j B_k \left(i \sum_\ell \epsilon_{jk\ell}\,\sigma_\ell + \delta_{jk}\mathbb{I}\right)$$

$$= \underbrace{i \sum_{jk\ell} \sigma_\ell A_j B_k \epsilon_{\ell jk}}_{\text{triple product } \vec{\sigma}\cdot(\vec{A}\times\vec{B})} + \underbrace{\sum_j A_j B_j}_{\vec{A}\cdot\vec{B}} \mathbb{I}$$

If $\hat{\vec{A}}$ and $\hat{\vec{B}}$ are QM vector operators then be sure to maintain the order

$$(\vec{\sigma}\cdot\vec{A})^2 = \underset{\substack{\uparrow \\ \vec{A}\cdot\vec{A} = |\vec{A}|^2}}{A^2}\,\mathbb{I}, \quad (\vec{\sigma}\cdot\underset{\substack{\uparrow \\ \text{unit vector}}}{\hat{u}})^2 = \mathbb{I}$$

$$e^{i\alpha(\vec{\sigma}\cdot\hat{u})} = \mathbb{I}\cos(\alpha) + i(\vec{\sigma}\cdot\hat{u})\sin(\alpha)$$

Prove with taylor series.

Euler identity for matrices

$$\exp(\underline{\underline{M}}) = \mathbb{I} + \underline{\underline{M}} + \frac{1}{2!}\underline{\underline{M}}\,\underline{\underline{M}} + \frac{1}{3!}(\underline{\underline{M}})^3 + \cdots$$

$\underline{\underline{M}}$ ↑ nxn matrix

like $e^{i\theta} = \cos\theta + i\sin\theta$

$$\sigma_x\sigma_y\sigma_z = i\mathbb{I}$$

Any 2x2 matrix even with complex entries can be expanded in the set $\{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}$

Also this set $\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$$\underline{\underline{M}} = \begin{pmatrix}m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix} = a_0\mathbb{I} + \vec{a}\cdot\vec{\sigma}$$

$$a_0 = \tfrac{1}{2}tr(\underline{\underline{M}}), \quad \vec{a} = \tfrac{1}{2}tr(\underline{\underline{M}}\vec{\sigma})$$

mean $a_j = \frac{1}{2}tr(\underline{\underline{M}}\sigma_j)$