For Spherical Tensor operators of rank-1 (vectors)

think of these as "spin-1" $\vec{V}$

Cf. Lecture 19 $[\hat{L}_i, \hat{V}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{V}_k$

Define $\hat{V}_\pm = \hat{V}_x \pm i \hat{V}_y$, $\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y$

$[\hat{L}_z, \hat{V}_z] = 0$, $[\hat{L}_z, \hat{V}_\pm] = \pm \hbar \hat{V}_\pm$

$[\hat{L}_\pm, \hat{V}_z] = \mp \hbar V_\pm$, $[\hat{L}_\pm, \hat{V}_\pm] = 0$

$[\hat{L}_\pm, V_\mp] = \pm 2\hbar V_z$

$$\langle n'l'm' | [\hat{L}_z, \hat{V}_+] | nlm \rangle = +\hbar \langle n'l'm' | \hat{V}_+ | nlm \rangle$$

$$= \langle n'l'm' | \hat{L}_z \hat{V}_+ | nlm \rangle - \langle n'l'm' | \hat{V}_+ \hat{L}_z | nlm \rangle$$

$$\Rightarrow [m' - (m+1)] \langle n'l'm' | \hat{V}_+ | nlm \rangle = 0$$

The matrix element will vanish unless $m' = m+1$

$\Delta m \equiv m' - m = +1$

Similarly $\langle n'l'm' | \hat{V}_z | nlm \rangle = 0$ unless $m' = m$, $\Delta m = 0$

$\langle n'l'm' | \hat{V}_- | nlm \rangle = 0$ unless $m' = m-1$, $\Delta m = -1$

$\Delta m = 0, \pm 1$

The other commutators give

$l_1 \quad l_2 = k \quad m_1 \quad m_2 = q$

$$\langle n'l'm'|\hat{V}_+|nlm\rangle = -\sqrt{2}\,\underbrace{\langle n'l'\|V\|nl\rangle}_{\text{reduced matrix element}}\,\underbrace{\langle l,\overset{k}{1},m,1|\overset{J}{l'}\overset{M}{m'}\rangle}_{\text{Clebsch-Gordan coefficient}}$$

$q=+1$

reduced matrix element $\alpha(n,n',l,l')$ but not m,m',q

$$\langle n'l'm'|\hat{V}_z|nlm\rangle = 1\,\langle n'l'\|V\|nl\rangle\langle l,1,m,0|l',m'\rangle$$

$q=0$

$$\langle n'l'm'|\hat{V}_-|nlm\rangle = +\sqrt{2}\,\langle n'l'\|V\|nl\rangle\langle l,1,m,-1|l',m'\rangle$$

$q=-1$ $\qquad \Delta l \equiv l'-l = 0, \pm 1$

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e.g. Find all the matrix elements of $\hat{\vec{r}}$ between hydrogen states $|n=2, l=1, m=0,\pm1\rangle$ and $|n'=3, l'=2, m'=0,\pm1,\pm2\rangle$

\# of matrix elements $5\times3\times3 = 45$!

Wigner-Eckart $\Rightarrow$ most of the 45 matrix elements are zero, only 9 are non-zero.

Only need to perform 1 integral.

Calculate one matrix element

$r^2 \sin\theta \, dr\, d\theta\, d\phi$

e.g. $\langle \overset{n'}{3}\,\overset{l'}{2}\,\overset{m'}{0} | \underset{q=0}{\hat{z}} | \overset{n}{2}\,\overset{l}{1}\,\overset{m}{0} \rangle = \int d^3r \; \psi^*_{320}(\vec{r}) \, [r\cos\theta] \, \psi_{210}(\vec{r}) = \frac{2^{12}3^3\sqrt{3}}{5^7} a_0$

$= \langle l_1 = 1, \underset{k}{l_2 = 1}, m_1 = 0, \underset{z \Leftrightarrow q=0}{m_2 = 0} | J = \underset{l'}{2}, M = \underset{m'}{0} \rangle \langle 32 \| V \| 21 \rangle$

$= \sqrt{\frac{2}{3}} \langle 32 \| V \| 21 \rangle$

$\Rightarrow$ Reduced matrix element $\langle 32 \| V \| 21 \rangle = \frac{2^{12}3^4}{\sqrt{2}\,5^7} a_0$

$\langle 321 | \hat{z} | 211 \rangle = \langle l_1 = 1, \underset{K}{l_2 = 1}, m_1 = 1, \underset{q=0}{m_2 = 0} | J = 2, M = 1 \rangle \langle 32 \| V \| 21 \rangle$

$= \frac{1}{\sqrt{2}} \left[\frac{2^{12}3^4}{\sqrt{2}\,5^7} a_0 \right]$

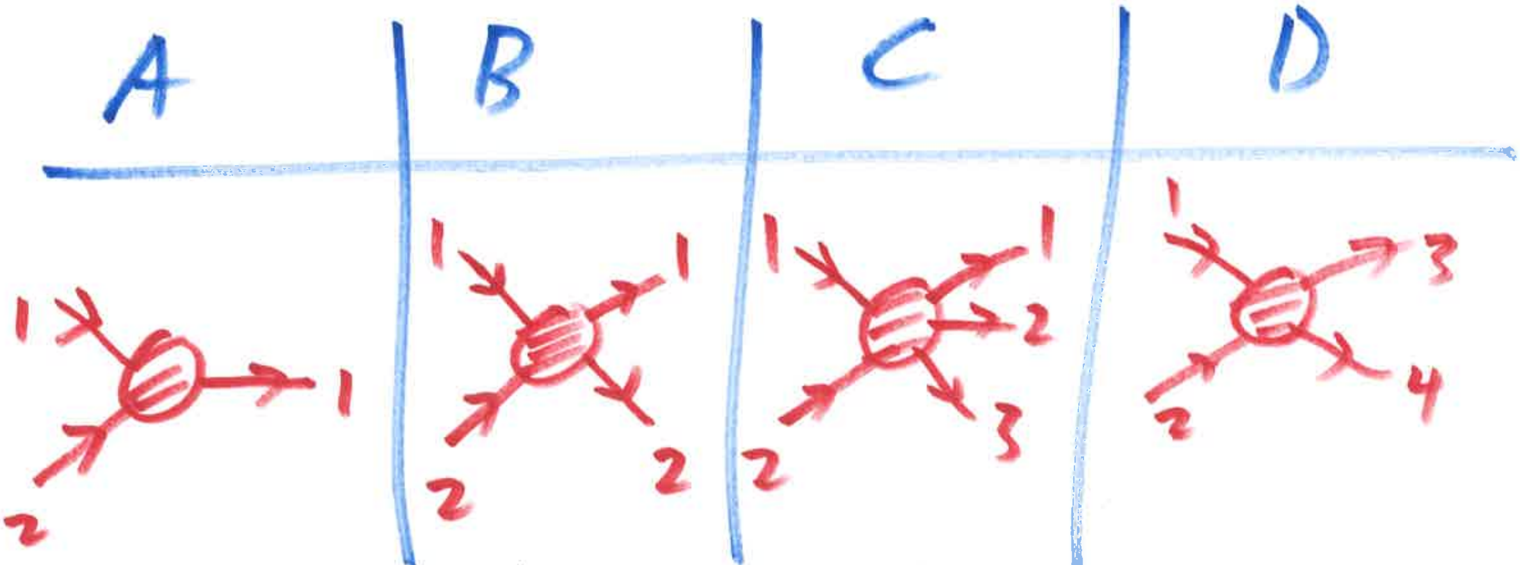
$\langle 321 | \hat{z} | 210 \rangle = 0$ since $\Delta m \neq 0$

$\langle \overset{JM}{3\,2\,2} | \underbrace{(\hat{x} + i\hat{y})}_{V_+,\; q=+1} | 2\,\overset{l_1}{1}\,\overset{m_1}{1} \rangle = -\sqrt{2} \langle l_1 = 1, \overset{=K}{l_2 = 1}, m_1 = 1, \overset{=q}{m_2 = 1} | J = 2, M = 2 \rangle \cdot \langle 32 \| V \| 21 \rangle$

$= -\sqrt{2}\,(1) \left[\frac{2^{12}3^4}{\sqrt{2}\,5^7} a_0 \right]$

C-G coeff.

Which of these are scattering?

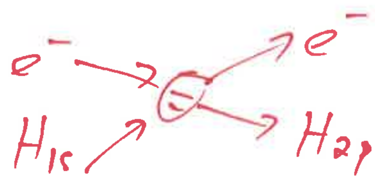


↑ Just this one

others are called "reactions"

elastic scattering → 1, 2 internal states do not change.

Not Elastic

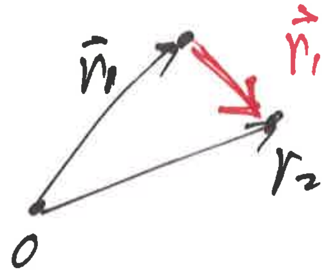


Scattering

Assumptions – no spin, no multiple scattering
e.g. no Bragg scattering from crystals.

Assume the potential energy $V(\vec{r}_1 - \vec{r}_2)$ depends only on the relative position of the particles 1, 2.

$\vec{r}_1 - \vec{r}_2 \equiv \vec{r}$ relative coordinate



With no assumptions, can always transform the two-body problem into a problem where one particle is ∞'ly heavy and the other has relative mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ (reduced mass)

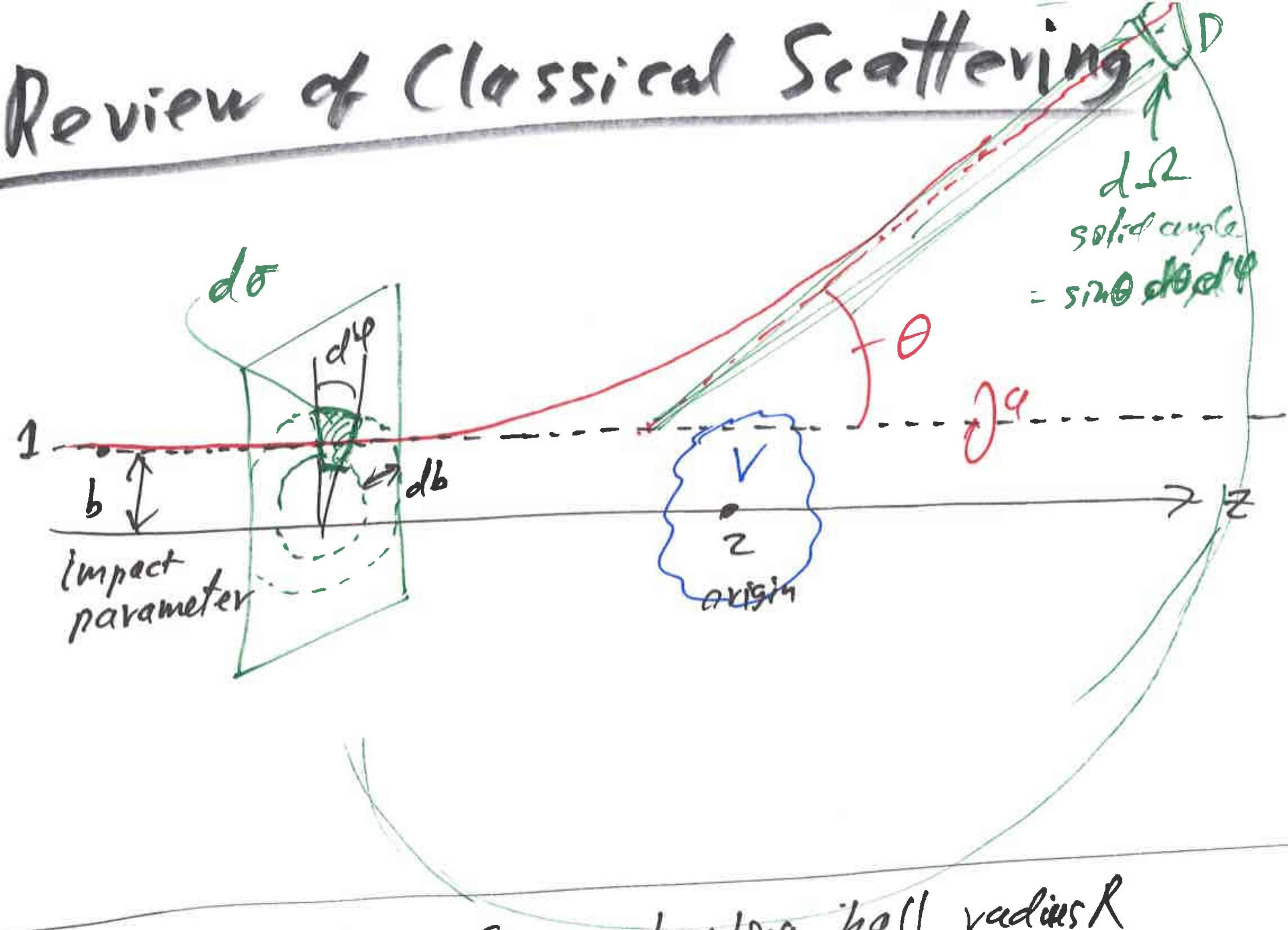
Change of variables $\{\vec{r}_1, \vec{r}_2\} \rightarrow \{\vec{R}_{cm}, \underset{\vec{r}_1 - \vec{r}_2}{\overset{\text{relative}}{\vec{r}}}\}$

$$\vec{R}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

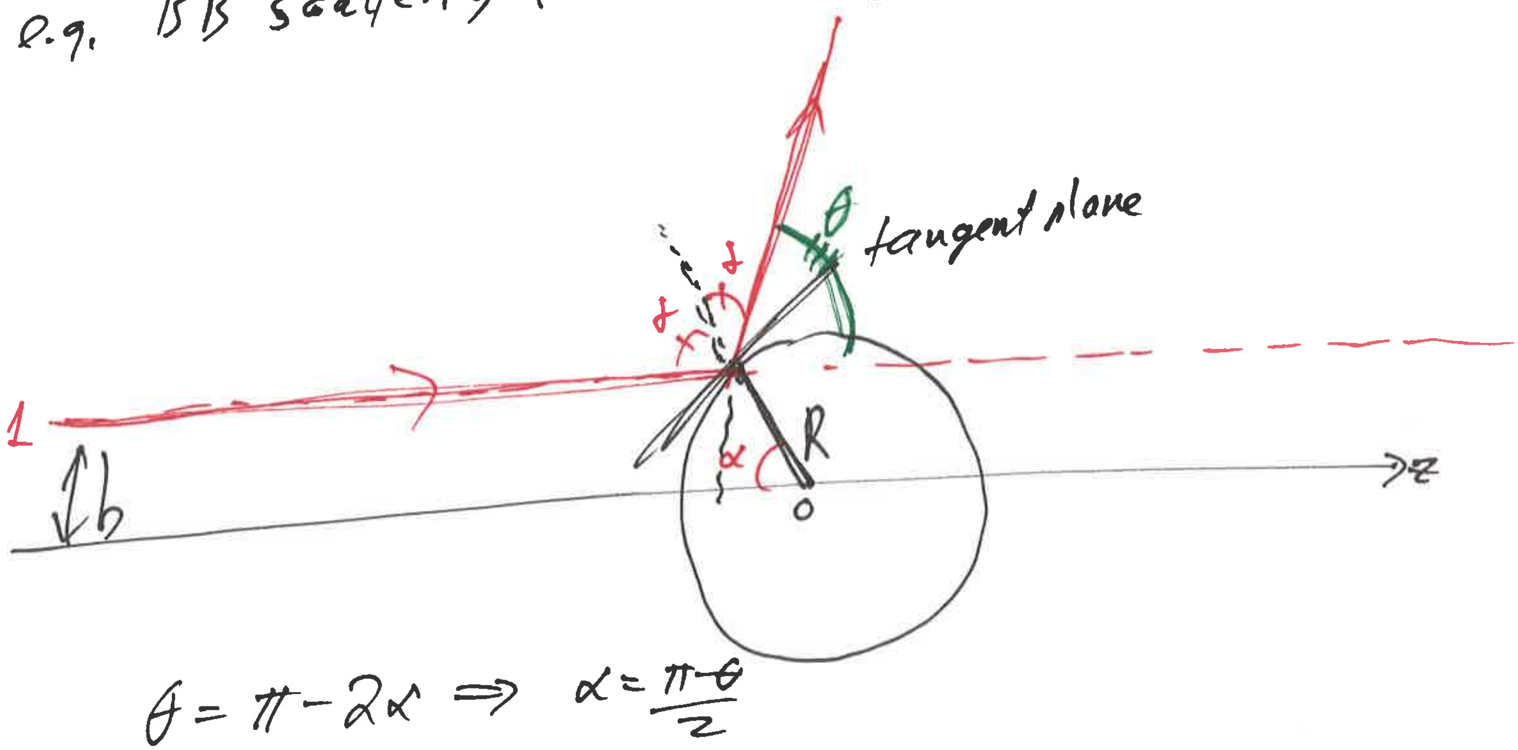
Often $V(r) \rightarrow$ central forces

We will not treat Coulomb potential $V(r) \sim \frac{1}{r}$ QM'ly
electrostatic force has ∞ range.

Review of Classical Scattering



e.g. BB scattering from a bowling ball radius R



$$\theta = \pi - 2\alpha \Rightarrow \alpha = \frac{\pi-\theta}{2}$$

$$b = R\sin\alpha = R\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right)$$

$$\theta = \begin{cases} 2\arccos\left(\frac{b}{R}\right), & b \le R \\ \emptyset, & b \ge R \end{cases}$$

$$d\sigma = b\, db\, d\varphi \qquad d\Omega = \sin\theta\, d\theta\, d\varphi$$

Differential scattering cross section

$$\sigma(\theta,\varphi) = \frac{d\sigma}{d\Omega} = \left|\frac{b\, db\, \cancel{d\varphi}}{\sin\theta\, d\theta\, \cancel{d\varphi}}\right| = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$\frac{db}{d\theta} = \frac{d}{d\theta}\left[R\cos\left(\frac{\theta}{2}\right)\right] = \frac{1}{2}R\sin\left(\frac{\theta}{2}\right)$$

$$\sigma(\theta,\varphi) = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{R\cos\left(\frac{\theta}{2}\right)}{\sin\theta}\frac{1}{2}R\sin\left(\frac{\theta}{2}\right) = \frac{R^2}{4}$$

Total cross section $\sigma = \iint \sigma(\theta,\varphi)\, d\Omega$

$$= \iint \frac{d\sigma}{d\Omega}\, d\Omega = \iint d\sigma = \pi R^2$$