e.g. QM scattering from a hard sphere.

$$V(r) = \begin{cases} \infty, & r \le R \\ 0, & r > R \end{cases}$$

central ✓ localized ✓

↑ $V(r)$ falls off faster than $\frac{1}{r^2}$

Boundary Condition $\psi(R,\theta,\varphi) = 0 \quad \forall \theta$

$$\psi(R) = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\left[\underbrace{j_\ell(kR)}_{\text{incoming}} + ik a_\ell \underbrace{h_\ell^{(1)}(kR)}_{\text{scattered}}\right] P_\ell(\cos\theta) = 0$$

$$\Rightarrow a_\ell = i \frac{j_\ell(kR)}{k h_\ell^{(1)}(kR)}$$

total cross section

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \left| \frac{j_\ell(kR)}{h_\ell^{(1)}(kR)} \right|^2 \quad \text{exact}$$

Look at low-energy scattering $kR \ll 1$; $\lambda \gg R$

$$\frac{j_\ell(kR)}{h_\ell^{(1)}(kR)} = \frac{j_\ell(kR)}{\underbrace{\cancel{j_\ell(kR)}}_{\text{small}} + i \underbrace{n_\ell(kR)}_{\text{big}}} \approx -i \frac{j_\ell(kR)}{n_\ell(kR)}$$

$$\approx \frac{-i\, 2^\ell \ell!\, (kR)^\ell / (2\ell+1)!}{-(2\ell)!\, (kR)^{-\ell-1} / (2^\ell \ell!)} = \frac{i}{2\ell+1} \left[\frac{2^\ell \ell!}{(2\ell)!}\right]^2 (kR)^{2\ell+1}$$

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[\frac{2^\ell \ell!}{(2\ell)!} \right]^4 (kR)^{4\ell+2}$$

$\ell = 0$ is largest term

s-wave ($\ell=0$)

$$\sigma \simeq 4\pi R^2$$ – 4x the geometric cross section

Phase shifts

Rayleigh formula

$$\psi_{inc}(\vec{r}) = A e^{ikz} = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

$$j_\ell(kr) = \frac{1}{2}\left[h_\ell^{(1)}(kr) + h_\ell^{(2)}(kr)\right] \xrightarrow[r\to\infty]{} \frac{1}{2kr}\left[(-i)^{\ell+1} e^{ikr} + (i)^{\ell+1} e^{-ikr}\right]$$

If $V(r) = 0$ (no scattering) then for large r

$$\psi(\vec{r}) = A \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2ikr} \left[e^{ikr} - (-1)^\ell e^{-ikr} \right] P_\ell(\cos\theta)$$

If $V(r) \neq 0$ $\quad e^{ikr} \to e^{i(kr+2\delta_\ell)}$ $\quad$ 2 is convention

δ_ℓ is the phase shift of the ℓ^{th} wave.

Relation between a_ℓ and δ_ℓ

$$a_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \frac{1}{k} e^{i\delta_\ell} \sin(\delta_\ell)$$

$$f(\theta) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin(\delta_\ell) P_\ell(\cos\theta)$$

no φ

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2(\delta_\ell)$$

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e.g. QM Hard Sphere scattering

$$a_\ell = \frac{i}{k} \frac{j_\ell(kR)}{h_\ell^{(1)}(kR)}$$

$$\delta_\ell = \arctan\left[\frac{j_\ell(kR)}{n_\ell(kR)}\right]$$

The First Max Born Approximation

If $V(\vec{r})$ is localized (goes to 0 faster than $\frac{1}{r^2}$ outside a finite radius).

Scattering Amplitude

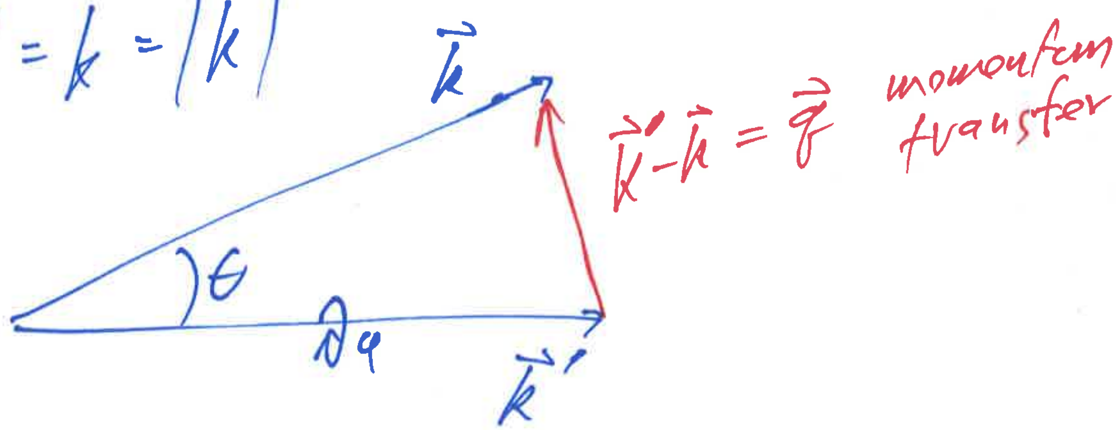
$$f(\theta,\varphi) \approx -\frac{m}{2\pi\hbar^2}\int_{\text{All space}} d^3r \; V(\vec{r})\, e^{i(\vec{k}'-\vec{k})\cdot\vec{r}}$$

Fourier Transform of potential

$\vec{k}'$ is incident wave vector $= k\hat{z}$
$\vec{k}$ is scattered wavevector $= k\hat{r}$

$\Big\}$ same magnitude $\rightarrow$ elastic scattering

$$|\vec{k}'| = k = |\vec{k}|$$



For low-energy scattering (small k, large λ)

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3r \; V(\vec{r})$$

e.g. Low-energy scattering from a soft sphere.

$$V(r) = \begin{cases} V_0 \,, & r<R \\ 0 \,, & r>R \end{cases}$$

$$f(\theta,\varphi) = -\frac{m}{2\pi\hbar^2} V_0 \left(\frac{4}{3}\pi R^3\right)$$ scattering amplitude

differential scattering cross section

$$\sigma(\theta,\varphi) = \frac{d\sigma}{d\Omega} = |f|^2 = \left(\frac{2mV_0R^3}{3\hbar^2}\right)^2$$

Total cross section

$$\sigma = \int d\Omega\, \sigma(\theta,\varphi) = 4\pi \left(\frac{2mV_0R^3}{3\hbar^2}\right)^2$$

Not Low-energy scattering, ~~but~~ central potential $V(\vec{r}) = V(r)$

$$f(\theta) \simeq -\frac{2m}{\hbar^2 k}\int_{r=0}^{\infty} r\, V(r) \sin(qr)\, dr$$

no φ

Yakawa scattering : $V(r) = \beta \dfrac{e^{-\mu r}}{r}$

$$f(\theta) = \frac{2m\beta}{\hbar^2(\mu^2+q^2)}$$

If $\beta = \dfrac{q_1 q_2}{4\pi\epsilon_0}$ and $\mu \to 0$ recover Rutherford (Coulomb scattering).