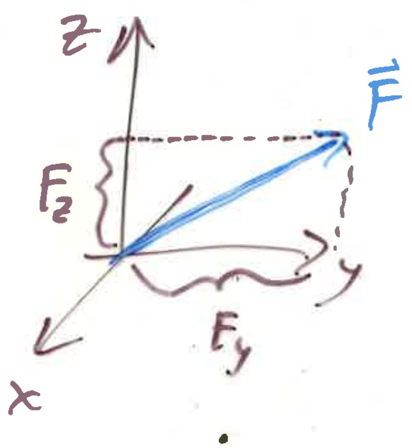
Fourier Series

Analogy with vector decomposition.



Expand $\vec{F}$ in terms of coefficients * basis vectors

Pick a set of basis vectors

e.g. $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$= \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

$= \{\hat{x}, \hat{y}, \hat{z}\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$

$$\vec{F} = \sum_{i=1}^{3} a_i \hat{e}_i = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

coefficients — basis vectors.

real number (in general, complex)

eg. $\vec{F} = 3\hat{e}_1 + 5\hat{e}_2 + 7\hat{e}_3$

Orthonormality:

Kronecker Delta

$\hat{e}_1 \cdot \hat{e}_1 = 1$

$\hat{e}_1 \cdot \hat{e}_2 = 0$ ... (9 total)

$\hat{e}_1 \cdot \hat{e}_3 = 0$ really just 6

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A different orthonormal basis set

$$\vec{u}_1 = (1, 0, -1) = 1\,\hat{e}_1 + 0\,\hat{e}_2 + (-1)\,\hat{e}_3$$

$$\vec{u}_2 = (1, 1, 1)$$

$$\vec{u}_3 = (1, -2, 1) \qquad \vec{u}_1 \cdot \vec{u}_1 = 2$$

Normalize

$$\hat{u}_1 = \frac{\vec{u}_1}{\sqrt{\vec{u}_1 \cdot \vec{u}_1}} \quad \text{so} \quad \hat{u}_1 \cdot \hat{u}_1 = \frac{\vec{u}_1 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = 1$$

$$\left\{ \begin{aligned} \hat{u}_1 &= \frac{1}{\sqrt{2}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \\ \hat{u}_2 &= \frac{1}{\sqrt{3}}(1, 1, 1) \\ \hat{u}_3 &= \frac{1}{\sqrt{6}}(1, -2, 1) \end{aligned} \right.$$

$$\vec{F} = \sum_{j=1}^{3} b_j \hat{u}_j = b_1 \hat{u}_1 + b_2 \hat{u}_2 + b_3 \hat{u}_3$$

Find the new coefficients $\{b_j\}$.

Use orthonormality: $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$

$$\hat{u}_i \cdot \vec{F} = \hat{u}_i \cdot \sum_{j=1}^{3} b_j \hat{u}_j = \sum_{j=1}^{3} b_j \, \hat{u}_i \cdot \hat{u}_j$$

$$= \sum_{j=1}^{3} b_j \, \delta_{ij} = b_i$$

$$b_1 = \hat{u}_1 \cdot \vec{F} = \frac{1}{\sqrt{2}}(1,0,-1) \cdot (3,5,7) = \frac{-4}{\sqrt{2}}$$

$$b_2 = \hat{u}_2 \cdot \vec{F} = \frac{1}{\sqrt{3}}(1,1,1) \cdot (3,5,7) = \frac{+15}{\sqrt{3}}$$

$$b_3 = \hat{u}_3 \cdot \vec{F} = \frac{1}{\sqrt{6}}(1,-2,1) \cdot (3,5,7) = 0$$

Check:

$$\vec{F} = \sum_{k=1}^{3} b_k \hat{u}_k$$

$$= \frac{-4}{\sqrt{2}} \frac{1}{\sqrt{2}}(1,0,-1) + \frac{15}{\sqrt{3}} \frac{1}{\sqrt{3}}(1,1,1) + \underbrace{\cancel{0 \frac{1}{\sqrt{6}}(1,-2,1)}}_{\to 0}$$

$$= (3,5,7)$$

Task: Find a basis set $\{\hat{v}_k\}$ such that $\vec{F} = c_1 \hat{v}_1 + 0\,\hat{v}_2 + 0\,\hat{v}_3$

$$\hat{v}_1 = \frac{1}{\sqrt{83}}(3,5,7)$$

$$\hat{v}_2 = \frac{1}{\sqrt{58}}(7,0,-3)$$

$$\hat{v}_3 = \hat{v}_1 \times \hat{v}_2$$

Completeness ~ Closure

(dual to orthonormality)

$$\vec{F} = \sum_{p=1}^{3} b_p \hat{u}_p \qquad b_p = \hat{u}_p \cdot \vec{F} = \vec{F} \cdot \hat{u}_p$$

$$\vec{F} = \sum_{p=1}^{3} (\vec{F} \cdot \hat{u}_p)\, \hat{u}_p = \vec{F} \cdot \left[\sum_{p=1}^{3} \underbrace{\hat{u}_p \hat{u}_p}_{\text{dyad notation}} \right]$$

sometimes $\hat{u}_p \wedge \hat{u}_p$

$\hat{u}_p \hat{u}_p$ is an outer product $\rightarrow$ matrix (3×3)

$\hat{u}_p \cdot \hat{u}_p$ is an inner product $\rightarrow$ scalar (1×1)

$\hat{u}_p \times \hat{u}_k$ is the cross product $\rightarrow$ vector (3×1)

Row and Column vectors

$$\vec{F} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \quad \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Dot product: $\hat{u}_1 \cdot \hat{u}_1 = \frac{1}{\sqrt{2}} \underbrace{(1, 0, -1)}_{1\times 3}{}^{*} \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{3\times 1} = 1$ $= 1\times 1$

Outer product: $\hat{e}_2 \wedge \hat{e}_1 = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{3\times 1} \underbrace{(1\ 0\ 0)^{*}}_{1\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$= 3\times 3$

~~Now with functions~~

$$\vec{F} = \vec{F} \cdot \sum_{p=1}^{3} \hat{u}_p \wedge \hat{u}_p = \vec{F} \cdot \mathbb{I}_{3\times 3}$$ ← identity matrix

check
$$\sum_{p=1}^{3} \hat{u}_p \wedge \hat{u}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \hat{u}_1 \wedge \hat{u}_1^T + \hat{u}_2 \wedge \hat{u}_2^T + \hat{u}_3 \wedge \hat{u}_3^T$$ ← transpose (hermitian conjugate in general)

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(1,0,-1) + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1,1,1) + \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}(1,-2,1)$$

$$= \underline{\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \end{pmatrix}} + \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \underline{\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & +4 & -2 \\ 1 & -2 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Longrightarrow \{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ spans 3-dimensional space

(all linearly independent)

Now with functions

Key concept: Hilbert Space

Normed vector space, possibly ∞ dimensional

- functional Analog of the dot product.
- A set of Basis Functions $\{u_1(t), u_2(t), \dots\}$

$$\hat{u}_i \cdot \hat{u}_j \longleftrightarrow \frac{1}{N}\int_{t_1}^{t_2} u_i^*(t)\, u_j(t)\, dt = \langle u_i(t) | u_j(t) \rangle$$

$$= \delta_{ij}$$

e.g. Taylor basis set of functions

$$\{1, x, x^2, x^3, \dots\}$$

e.g. Hermite polynomials, Laguerre, Laplace, ...

e.g. Fourier basis set of functions

$$\left\{ \overset{\cos(0\omega t)}{1}, \cos(\omega t), \cos(2\omega t), \cos(3\omega t), \dots \atop \sin(\omega t), \sin(2\omega t), \sin(3\omega t), \dots \right\}$$

How to chose ω, the fundamental frequency?

$\omega = \frac{2\pi}{T}$ where T is the period.

$$\hat{u}_1 = \frac{1}{\sqrt{2}}(1,0,-1)$$

$$\hat{u}_2 = \frac{1}{\sqrt{3}}(1,1,1)$$

$$\hat{u}_3 = \frac{1}{\sqrt{6}}(1,-2,1)$$

Check for closure

outer product

$$\sum_{q=1}^{3} \hat{u}_q \hat{u}_q = \mathbb{I}_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{u}_1 \hat{u}_1 \quad + \quad \hat{u}_2 \hat{u}_2 \quad + \quad \hat{u}_3 \hat{u}_2$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (1,0,-1) + \left(\frac{1}{\sqrt{3}}\right)^2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (1,1,1) + \left(\frac{1}{\sqrt{6}}\right)^2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} (1,-2,1)$$

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

Quantum Mechanics

Non-relativistic QM, no spin, massive

Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t} \Psi(x,y,z,t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi_{(x,y,z,t)} + V_{(x,y,z)} \Psi_{(x,y,z,t)}$$

$\psi_{(x,y,z)}$ space only

Relativistic

spin-0 : Klein-Gordon

spin-$\frac{\hbar}{2}$: Dirac Equation

spin-$\hbar$: Proca-Maxwell

spin-$\frac{3}{2}\hbar$: Rarita-Schwinger

spin-$2\hbar$: (Massless) Einstein

Guess: $\Psi = e^{\frac{i}{\hbar}(x \cdot p_x - y \cdot p_y + z \cdot p_z + Et)}$

$$\frac{\partial}{\partial t} \Psi = \frac{iE}{\hbar} \Psi$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z}$$

$$\nabla^2 \Psi = \frac{i}{\hbar}(p_x^2 + p_y^2 + p_z^2)$$

$$\boxed{E = \frac{p^2}{2m} + V}$$

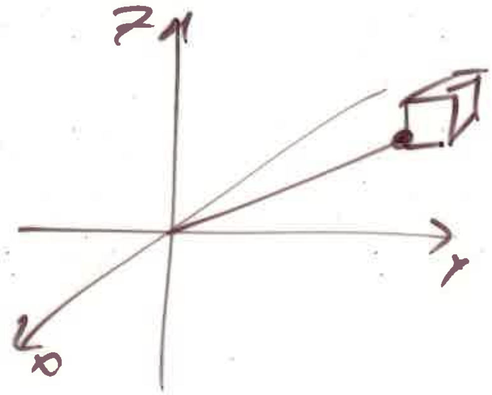
Max Born interpret

$\Psi_{(x,y,z,t)}$ is the probability amplitude for finding the particle.

complex

$$\underbrace{\Psi^*_{(x,y,z,t)}\,\Psi_{(x,y,z,t)}\,dx\,dy\,dz}$$

probability of finding the particle at (x,y,z) within $dx\,dy\,dz$



One dimension

$\Psi^*_{(x)}\,\Psi_{(x)}\,dx$ is the probability of finding the particle between x and $x+dx$

What is the probability (in one dimension) of finding the particle between $x=a$, and $x=b$?

$$\int_a^b \Psi^*_{(x)} \Psi_{(x)} \, dx$$

What is the probability of finding the particle anywhere

$$\int_{-\infty}^{+\infty} \Psi^*_{(x)} \Psi_{(x)} \, dx = 1$$

Differential Equations

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V \Psi$$

2^{nd} order, linear (in Ψ), Partial, [redacted]-homogenous

differential equation