Number Operator $\quad \hat{n} \equiv \hat{a}_+ \hat{a}_- = n$

$$\hat{n}\psi_n = \hat{a}_+(\hat{a}_-\psi_n) = n\psi_n$$

If $n=0$, $\quad \hat{n}\psi_0 = \hat{a}_+(\cancelto{0}{\hat{a}_-\psi_0}) = 0\,\psi_0 = n\,\psi_0$

If $n>0$, suppose know ψ_n normalized.

$$\hat{a}_-\psi_n \propto \psi_{n-1} \;, \quad \hat{a}_+(\hat{a}_-\psi_n) \propto \psi_n$$

$$\hat{n}\psi_n = \hat{a}_+\hat{a}_-\psi_n = ? \quad \text{expect } n\psi_n$$

$$\hat{H}\psi_n = \hbar\omega\left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right)\psi_n = E_n\psi_n = \left(n+\tfrac{1}{2}\right)\hbar\omega\,\psi_n$$

$$\Rightarrow \left(\hat{a}_+\hat{a}_- + \cancel{\tfrac{1}{2}}\right)\psi_n = \left(n+\cancel{\tfrac{1}{2}}\right)\psi_n \Rightarrow \hat{a}_+\hat{a}_-\psi_n = \hat{n}\psi_n = n\psi_n$$

$$\hat{H}\psi_n = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \tfrac{1}{2}\right)\psi_n = E_n\psi_n = \left(n+\tfrac{1}{2}\right)\hbar\omega\,\psi_n$$

$$\hat{a}_-\hat{a}_+\psi_n = (n+1)\psi_n$$

We have $\Psi_0(x,t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{m\omega x^2}{2\hbar}\right] \exp\left[\frac{-it}{\hbar}\left(\tfrac{1}{2}\right)\hbar\omega\right]$

Normalized

Mathematical Induction:

Assume we have ψ_n, normalized $\langle \psi_n | \psi_n \rangle = 1$

dot product, like $\hat{u}_n \cdot \hat{u}_n = 1$

What is:

$$\langle (\hat{a}_+ \psi_n) | \hat{a}_+ \psi_n \rangle = \langle \psi_n | (\hat{a}_+)^\dagger \hat{a}_+ \psi_n \rangle$$

$$= \langle \psi_n | \hat{a}_- \hat{a}_+ | \psi_n \rangle$$

$$= \langle \psi_n | (n+1) | \psi_n \rangle$$

$$= (n+1) \langle \psi_n | \psi_n \rangle = (n+1)$$

transpose complex conjugate $(\hat{a}_+ \psi_n)^\dagger$

$(AB)^\dagger = B^\dagger A^\dagger$

$$\Longrightarrow \frac{| \hat{a}_+ \psi_n \rangle}{\sqrt{n+1}} \text{ is normalized.}$$

$$\langle (\hat{a}_+ \psi_n) | \hat{a}_+ \psi_n \rangle = \int_{x=-\infty}^{\infty} [\hat{a}_+ \psi_n(x)]^\dagger \hat{a}_+ \psi_n(x) \, dx$$

$$= \int_{x=-\infty}^{+\infty} \psi_n(x)^\dagger (\hat{a}_+)^\dagger \hat{a}_+ \psi_n(x) \, dx = \langle \psi_n | \hat{a}_+^\dagger \hat{a}_+ \psi_n \rangle$$

$$\langle \psi_n | \hat{H} | \psi_n \rangle = \langle \psi_n | (\hat{H} \psi_n) \rangle = \langle \psi_n | E_n \psi_n \rangle$$

$$= E_n \langle \psi_n | \psi_n \rangle = E_n$$

$$= \langle \hat{H} \psi_n | \psi_n \rangle = (\langle \psi_n | \hat{H}^\dagger) | \psi_n \rangle \Longleftrightarrow \hat{H} = \hat{H}^\dagger$$

$\hat{H}$ is hermitian $\Longrightarrow$ eigenvalues are real

$|\psi_0\rangle$ ground state is normalized

$$\hat{a}_+|\psi_0\rangle = \frac{|\psi_1\rangle}{\sqrt{0+1}} = |\psi_1\rangle$$

$$\hat{a}_+|\psi_1\rangle \propto |\psi_2\rangle \Longrightarrow |\psi_2\rangle = \frac{\hat{a}_+|\psi_1\rangle}{\sqrt{2}} = \frac{\hat{a}_+^2|\psi_0\rangle}{\sqrt{2}}$$

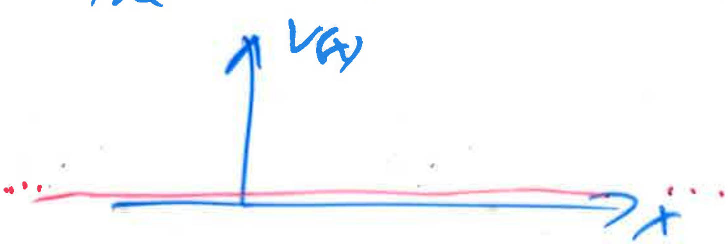
$\vdots$

$$|\psi_n\rangle = \frac{\hat{a}_+^n|\psi_0\rangle}{\sqrt{n}\sqrt{n-1}\sqrt{n-2}\cdots\sqrt{1}} = \frac{\hat{a}_+^n|\psi_0\rangle}{\sqrt{n!}}$$

$$\hat{a}_-|\psi_n\rangle = \sqrt{n}\,|\psi_{n-1}\rangle$$

$$\hat{a}_+|\psi_n\rangle = \sqrt{n+1}\,|\psi_{n+1}\rangle$$

The Free Particle — $V(x) =$ ~~constant~~ 0 by choice. everywhere



$$\Psi(x,t) = f(x)\cdot g(t)$$

separation constant

S.E. $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} f(x) = E\, f(x)$

$$f''(x) = -\left(\frac{2mE}{\hbar^2}\right) f(x) = -k^2 f(x)$$

time part

$g(t) = g_0\, e^{-\frac{itE}{\hbar}}$

$= g_0\, e^{-i\omega t}$

$\Rightarrow \omega = E/\hbar$

$$f(x) = C\cos(kx) + D\sin(kx)$$

$k \equiv \sqrt{\frac{2mE}{\hbar^2}}$ has dimensions of L^{-1}

will be the wavenumber $= \frac{2\pi}{\lambda}$

$$f(x) = A e^{ikx} + B e^{-ikx}$$

complex

Euler

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Psi(x,t) = f(x)\cdot g(t) = A e^{i(kx-\omega t)} + B e^{-i(kx+\omega t)}$$

plane waves

right-going $\longrightarrow$

$\longleftarrow$ left-going

No boundaries (walls) $\Rightarrow$ k is not quantized

e.g. ∞ square well $k_n = \frac{n\pi}{a}$, $n = 1, 2, 3, \ldots$

$\Rightarrow$ Energy E not quantized

e.g. ∞ square well $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, SHO: $E_n = (n+\frac{1}{2})\hbar\omega$

$\Rightarrow$ Ψ wiggles everywhere, never exponentially damped

e.g. SHO

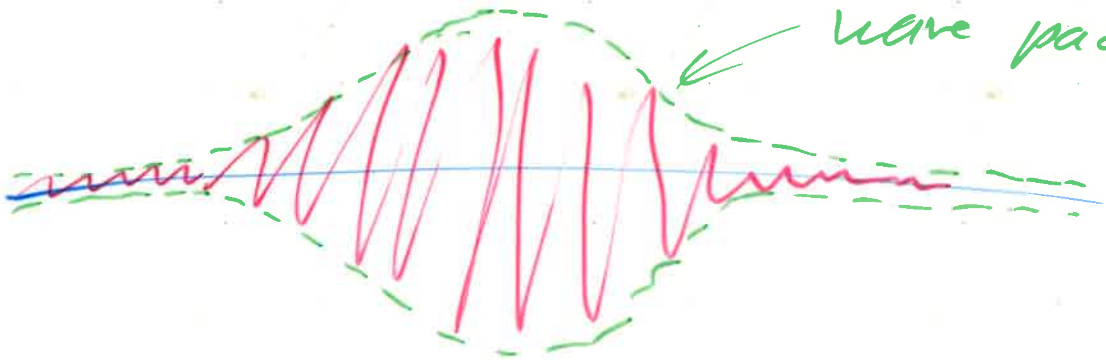
$\Rightarrow$ $\int_{-\infty}^{\infty} |\Psi|^2 dx = \infty$ $\Rightarrow$ plane waves can not be normalized.

$\Rightarrow$ We know the energy exactly $\Rightarrow \Delta E = 0$

$\Rightarrow$ therefore we know nothing about position

$\Delta x = \infty \quad (\Delta p = 0)$ Heisenberg Uncertainty

Plane waves can not describe massive particles by themselves. $\Rightarrow$ Combine different k's (E's) to localize a particle



Need $\Delta E \neq 0$, $\Delta p \neq 0 \Rightarrow$ to get $\Delta x \neq \infty$

Use $\underset{\longrightarrow}{e^{i(kx - \omega t)}}$, $\underset{\longleftarrow}{e^{-i(kx + \omega t)}}$ as basis vectors in the Hilbert space.

∞ square well, SHO $\Rightarrow$ infinite number of basis vectors, but countably infinite $\aleph_0$ (aleph-null)

Free particle $\Rightarrow$ non-countably infinite # of basis vectors, $\aleph_1$ (aleph one)