Radial Equation

no θ, or φ dependence.

$$\frac{d}{dr}\left[r^2 \frac{dR(r)}{dr}\right] - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]R(r) = l(l+1)R(r)$$

For $E<0$, there will be bound states, E is quantized, discrete, labelled by $\underline{l}, \underline{n}$.

For $E>0$, there will be a continuum of energy scattering states. E not quantized.

e.g. Infinite spherical well

$$V(r) = \begin{cases} 0, & r<a \\ \infty, & r>a \end{cases}$$

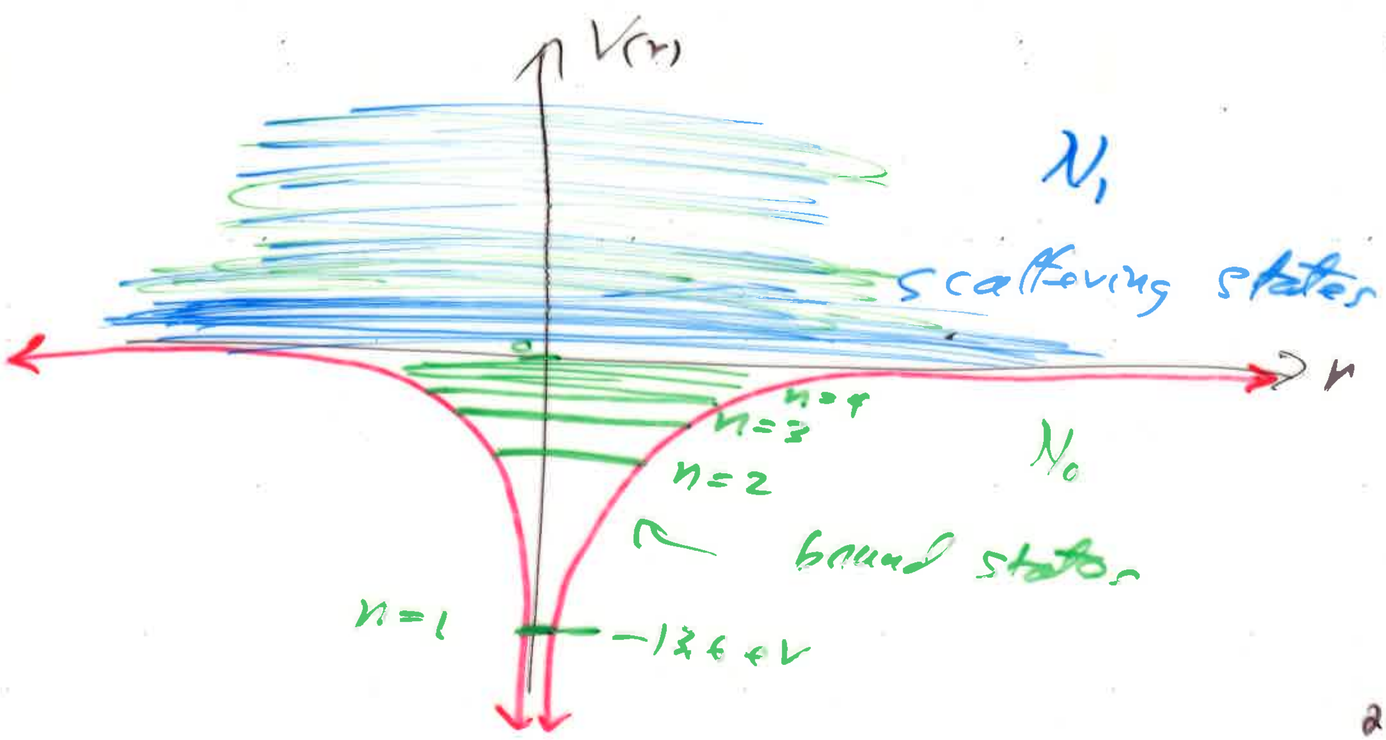
E_{nl} (not m)

e.g. H atom

$$V(r) = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$$

E_n (not l, m)

n = principal quantum #



Change variables $\quad R(r) \equiv \dfrac{u(r)}{r}$

$$\frac{dR(r)}{dr} = \frac{1}{r}\frac{du(r)}{dr} - \frac{u(r)}{r^2}$$

$$\frac{d}{dr}\left[r^2 \frac{dR(r)}{dr}\right] = \frac{d}{dr}\left[r\,u'(r) - u(r)\right]$$

$$= r\,u''(r) + \cancel{\left(\frac{dr}{dr}\right) u'(r) - u'(r)} = r\,u''(r)$$

$$\longrightarrow -\frac{\hbar^2}{2m}u''(r) + \left[V(r) + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right]u(r) = E\,u(r)$$

Looks like one-dimensional S.E.

but with an effective potential

Aside

$$V_{eff}(\vec{r})$$

$$= mgz - \tfrac{1}{2}m\omega^2 s^2$$

$$\vec{F} = -\vec{\nabla} V_{eff}(\vec{r})$$

$$= -mg\,\hat{e}_z + m\omega^2 s\,\hat{e}_s$$

Normalization

$$\int_{r=0}^{\infty} R^*_{(r)} R_{(r)}\, r^2\, dr = 1 = \int_{r=0}^{\infty} u^*_{(r)}\, u(r)\, dr$$

For Hydrogen

Look at bound states: $E<0$

S.E. $\frac{-\hbar^2}{2m} u''(r) + \left[\frac{-e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right] u(r) = E\, u(r)$

← mass ← mass

Find E_n eigenvalues and $u_{n\ell}(r)$ functions.

$$E_n = -\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2} = E_1 \frac{1}{n^2} = (-13.6\,eV)\frac{1}{n^2}$$

m_e = electron mass $e = 1.6\times10^{-19}C$ proton charge

Bohr radius $a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2} = 0.529\times10^{-10}\,m = \frac{1}{2}\text{Å}$

We assumed the proton was infinitely heavy and did not move. Now fix this.

Instead of m_e, use $\mu = \frac{m_p m_e}{m_p + m_e}$ reduced mass

$\mu < m_p$, $\mu < m_e$

$m_p = 1836.15267389\, m_e$

$$\mu = \left(\frac{m_p}{m_p+m_e}\right) m_e = \left(\frac{1836}{1836+1}\right) m_e = 0.99945\, m_e$$

For positronium

$e^+ \; e^-$

$$\mu = \frac{m_{e^+} m_{e^-}}{m_{e^+} + m_{e^-}} = \frac{m_e^2}{2m_e} = \frac{1}{2} m_e$$

H-atom

principal q.n. : $n = 1, 2, 3, \ldots$ infinitely many

azimuthal
(orbital) q.n: $\ell = 0, 1, 2, 3 \ldots (n-1).$
$\quad\quad\quad\quad\quad\quad s \quad p \quad d \quad f$

magnetic q.n. : $m = \underbrace{-\ell, -\ell+1, \ldots 0, \ldots \ell-1, \ell}_{2\ell+1}.$

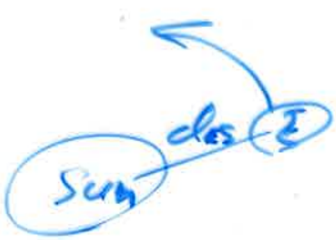
Degeneracy d

$n=1 \Rightarrow \ell=0,\ m=0$ 1 state $d=1$

$n=2 \cdot \begin{cases} \ell=1, & m=+1,0,-1 \\ \ell=0, & m=0 \end{cases}$ 4 states $d=4$

$n=3 \begin{cases} \ell=2, & m=+2,+1,0,-1,-2 \\ \ell=1, & m=+1,0,-1 \\ \ell=0, & m=0 \end{cases}$ 9 states $d=9$

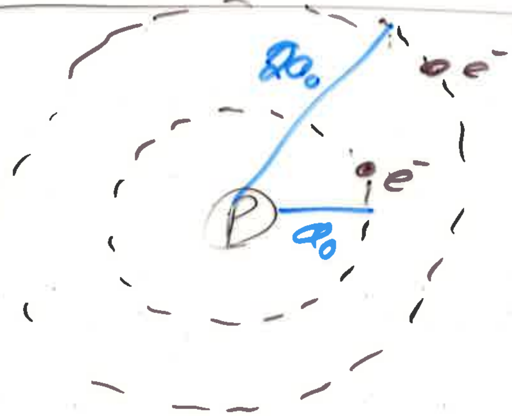
n $\quad d = 2n^2$ for spin $\frac{1}{2}$ $\quad d = n^2$



orbital angular momentum
$L = M_E v d_{ES}$

spin angular momentum
$S = \frac{2}{5} M_E R_E^2 \omega \quad \omega = \frac{2\pi}{1\,day}$

Old Quantum Mechanics



Lowest level does have orbital angular momentum.

New QM Lowest level $\ell = 0$ no angular momentum.

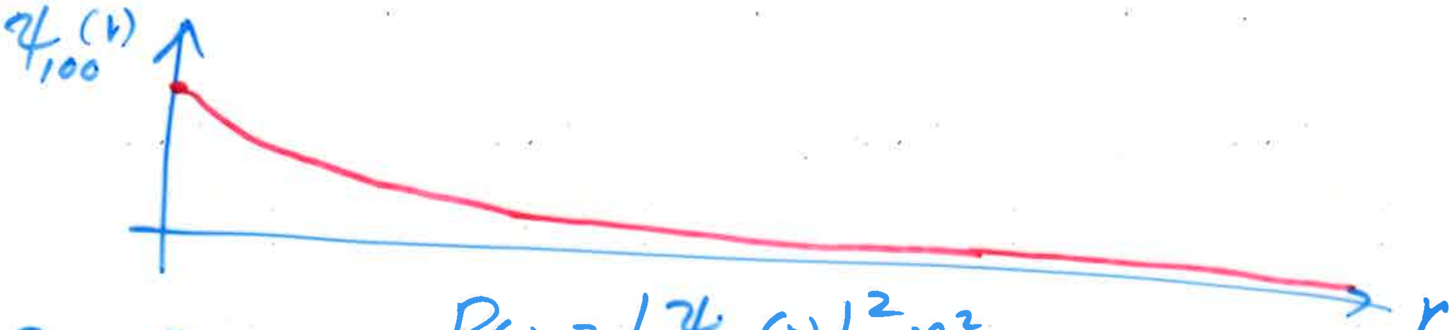
$$f(r,\theta,\varphi) = R(r)\underbrace{T(\theta)\,F(\varphi)}_{Y(\theta,\varphi)} = \psi(r,\theta,\varphi)$$

$$\Psi(r,\theta,\varphi,t) = \psi\, g(t)$$

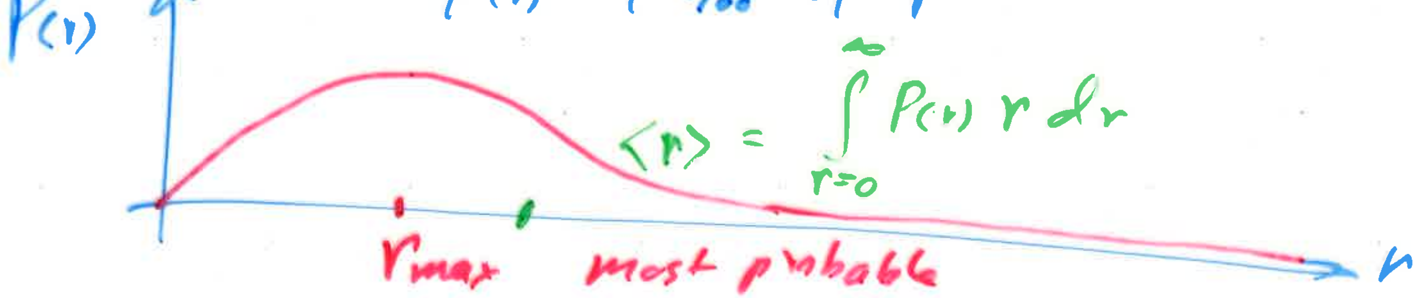
$$\psi_{n\ell m}(r,\theta,\varphi)$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$\psi_{100}(r,\theta,\varphi) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-\frac{r}{a_0}} = \frac{1}{\pi^{1/2} a_0^{3/2}}\, e^{-r/a_0}$$



$$P(r) = |\psi_{100}(r)|^2 r^2$$



$$\langle r \rangle = \int_{r=0}^{\infty} P(r)\, r\, dr$$