The ground state & the Uncertainty Relations for the H.O.

First the Virial Theorem:

$$\frac{d}{dt}\langle \hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$$
$$= 0 \quad \text{for stationary states}$$

Claim: $$2\langle T\rangle = \langle \vec{r}\cdot\vec{\nabla}V(r)\rangle$$

For stationary states of the H.O.

Proof $$\frac{d}{dt}\langle \vec{r}\cdot\vec{p}\rangle = 0 \qquad \text{"For Stationary State"}$$
$$= \frac{i}{\hbar}\langle[\hat{H},\vec{r}\cdot\vec{p}]\rangle$$

$$\Rightarrow \langle[\hat{H},\vec{r}\cdot\vec{p}]\rangle = 0$$

$$[\hat{H},\vec{r}\cdot\vec{p}] = \left[\frac{p^2}{2m},\vec{r}\cdot\vec{p}\right] + [V(r),\vec{r}\cdot\vec{p}]$$

$$= \frac{1}{2m}[p^2, r_i]p_i + r_i\underbrace{[V(r),p_i]}_{i\hbar\frac{\partial}{\partial x_i}V(r)}$$

$$[p^2, r_i] = p_\ell[p_\ell, r_i] + [p_\ell, r_i]p_\ell = -2i\hbar p_i$$

$$\Rightarrow \left\langle -i\hbar\frac{p^2}{m}\right\rangle + \langle i\hbar\,\vec{r}\cdot\vec{\nabla}V\rangle = 0$$

$$2\langle T\rangle = \langle \vec{r}\cdot\vec{\nabla}V\rangle \qquad \text{Q.E.D.}$$
(For stationary states)

For the 1-d H.O.: $$V(x) = \frac{1}{2}m\omega^2x^2$$
$$\vec{r}\cdot\vec{\nabla}V = x\frac{\partial}{\partial x}V = m\omega^2x^2 = 2V(x)$$

$$\Rightarrow \langle T\rangle = \langle V\rangle$$

$$\hat{H} = T + V \Rightarrow \langle\hat{H}\rangle = 2\langle T\rangle$$
$$= 2\langle V\rangle$$

$$\Rightarrow \langle T\rangle = \frac{1}{2}\hbar\omega(n+1/2)$$

$$\langle V \rangle = \tfrac{1}{2}\hbar\omega\left(n+\tfrac{1}{2}\right)$$

$$T = \frac{P^2}{2m} \quad \Rightarrow \quad \langle P^2 \rangle = \hbar m\omega\left(n+\tfrac{1}{2}\right)$$

$$V(x) = \tfrac{1}{2}m\omega^2 X^2 \qquad \langle X^2 \rangle = \frac{2}{m\omega^2}\langle V \rangle = \frac{\cancel{2}}{m\omega^{\cancel{2}}}\,\frac{1}{\cancel{2}}\,\hbar\cancel{\omega}\left(n+\tfrac{1}{2}\right)$$

$$\langle X^2 \rangle = \frac{\hbar}{m\omega}\left(n+\tfrac{1}{2}\right)$$

Also, $\left.\begin{aligned}\langle P \rangle &= 0 \\ \langle X \rangle &= 0\end{aligned}\right\}$ since $\psi_n(-x) = (-1)^n \psi_n(x)$

$$|\psi_n(-x)|^2 = |\psi_n(x)|^2$$

$$(\Delta P\,\Delta X)^2 = \langle X^2 \rangle \langle P^2 \rangle$$

$$= \frac{\hbar}{\cancel{m\omega}}\left(n+\tfrac{1}{2}\right)\hbar\cancel{m\omega}\left(n+\tfrac{1}{2}\right)$$

$$\Rightarrow \quad \Delta P\,\Delta X = \hbar\left(n+\tfrac{1}{2}\right)$$

"for stationary states of the H.O."

$$\therefore \quad (\Delta X\,\Delta P)_{min} = \frac{\hbar}{2}$$

"minimum uncertainty product"

But $n=0 \Rightarrow$ The gnd state:

$$\psi_0(x) = A_0\, e^{-\xi^2/2}$$

"gaussian in X"

$$E_0 = \frac{\hbar\omega}{2}$$

"Zero Point Energy"

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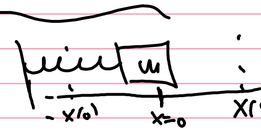
Correspondence w/ the Classical H.O.



$$V(x) = \tfrac{1}{2}KX^2, \quad E = \frac{P^2}{2m} + V(x)$$

$$F = -KX \;; \quad \omega = \sqrt{K/m}$$

$$= m\frac{d^2x}{dt^2}$$

$$\Rightarrow \quad X(t) = X(0)\cos(\omega t)$$

$$E = \tfrac{1}{2}K X^2(0) = \tfrac{1}{2}m\omega^2 X_0$$

$$X_0 = \sqrt{\frac{2E}{m\omega^2}}$$

$$= \text{constant in time}$$

Probability distribution in X : clearly $\rho(-x) = \rho(x)$

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$$\rho(x)\,dx = \rho(t)\,dt \qquad \rho(t) = \text{constant} = \frac{1}{T}$$

$$\rho(x)\,dx = \frac{1}{T}\,dt$$

$$\rho(x) = \frac{1}{T}\frac{1}{\frac{dx}{dt}} = \frac{1}{T\dot{X}(t)}$$

$$\dot{X}(t) = -X(0)\,\omega\cos\omega t \qquad \dot{X}^2 = X_0^2\omega^2\cos^2\omega t$$

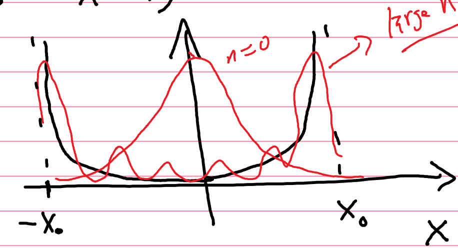
$$= X_0^2\omega^2(1 - S^2\omega t)$$

$$= (X_0^2 - X^2(t))\,\omega^2$$

$$\dot{X}(t) = \sqrt{X_0^2 - X^2(t)}$$

$$T = \frac{2\pi}{\omega}$$

$$\rho(x) = \omega\frac{1}{T X_0}\frac{1}{\left[1 - \left(X(t)/X_0\right)^2\right]^{1/2}}$$



$$\rho(x) = \frac{1}{2\pi X_0}\frac{1}{\left[1 - \left(\frac{X(t)}{X_0}\right)^2\right]^{1/2}}$$

In Q.M. $\rho(x) = |\psi_n(x)|^2 = \frac{A_0^2}{2^n n!}\left(H_n(\xi)\right)^2 e^{-\xi^2}$; $\xi = \sqrt{m\omega}$

"nth order polynomial"

The classical limit is obtained in the limit $n \to \infty$

$\hbar\omega$ $\hbar\omega/2$ $E_n = \hbar\omega(n + 1/2)$

Well, not quite. One must also work w/ a superposition of stationary states. One single stationary state will not yield classical Limit! the particular superposition which does is called a Coherent State.