DELTA-FUNCTION POTENTIALS IN TWO- AND THREE-DIMENSIONAL QUANTUM MECHANICS*

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ABSTRACT

Two- and three-dimensional δ-function interactions in Schrödinger theory are the formal non-relativistic limits for the scalar field φ^4 self-interactions of relativistic quantum field theory in $(2+1)$- and $(3+1)$-dimensional space-time, respectively. The quantum mechanical problems possess non-trivial dynamics if infinite renormalization or self-adjoint extension of the Hamiltonians is performed. The field theory is known to exist for the lower dimensionality, but for the higher dimensionality it is conjectured to be trivial. Thus the non-relativistic limits, supplemented by renormalization or self-adjoint extension, do not show this variety. Also the planar δ-function interaction formally admits an $SO(2,1)$ dynamical symmetry, but quantization necessarily spoils the invariance, putting into evidence the simplest example of quantum mechanical symmetry breaking. In this pedagogical essay, dedicated to the memory of M. A. B. Bég, work initiated by him is elaborated.

In Memoriam M. A. B. Bég

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I. INTRODUCTION

Baqi Bég was an eminent particle theorist who never lost sight of the *physical* goals of our profession — so much more difficult to attain than the purely mathematical. Nevertheless, he used mathematical tools with ease, and at various times during the development of modern fundamental physics Bég illuminated crucial phenomenological/experimental issues. The importance of Baqi's work and the esteem in which it is held are well exemplified by "Bég's Theorem" of nuclear physics, which "surprised" R. Peierls and figures in his memoirs;[1] "Bég's Sum Rule" of current algebra, discussed in S. Adler's and R. Dashen's definitive book on that subject;[2] his many contributions to the quark model, through the framework of higher symmetries, a selection of which is collected in F. Dyson's reprint volume;[3] and also his attempts to complete the "standard model" by urging a dynamical mechanism for its spontaneous symmetry breaking, as documented in the collection of sources for these ideas.[4]

Spontaneous symmetry breaking in the standard model concerned Bég in the final period of his research. The tension between the model's unquestioned phenomenological success and the theoretical inadequacy of the scalar field (Higgs) mechanism for its spontaneous symmetry breaking informed his activity. Taking account of the conjectured non-existence of the scalar field self-interaction in four-dimensional space-time,[5] Baqi on the one hand attempted to do away with the Higgs sector of the model, replacing its function by a dynamical mechanism;[4] on the other hand he tried, characteristically, to extract phenomenologically useful and experimentally verifiable information from the conjectured triviality of Higgs field dynamics.[6]

In order to understand better the nature of the scalar field φ^4 self-interaction Bég and Furlong[7] had the good idea to consider the non-relativistic limit of the model. When that limit is taken formally, particle number is conserved, the interaction between particles becomes the zero-range δ-function and dynamics is governed by tractable quantum mechanics. But even though one is dealing with quantum mechanics, there arise ultraviolet divergences, reminiscent of quantum field theory and renormalization is required. Bég and Furlong then showed that the δ-function interaction in three-dimensional space gives rise to a trivial S-matrix, when the bare/unrenormalized coupling constant is finite, a result that does not disagree with triviality of the relativistic theory, but of course neither does it establish triviality relativistically. The same conclusion was later obtained by K. Huang[8] in an independent investigation.

Owing to the distress inflicted on the standard model by the absence of the Higgs interaction, various proposals have been made for defining a non-trivial relativistic φ^4 theory. While one cannot deem these attempts convincingly successful, it is natural to inquire whether one can evade the Bég–Furlong–Huang result in the non-relativistic theory.

In this essay, which is inspired by Baqi's work and is dedicated to his memory, I point out that indeed procedures are available for defining a non-trivial δ-function interaction in three dimensions. Moreover, the same methods work in the same way in two dimensions; indeed they *must* be used if triviality is to be avoided there, because also the planar δ-function interaction needs renormalization, which extinguishes a finite bare/unrenormalized coupling.

While the positive non-relativistic results are pleasing, they do not of course illuminate the situation in the relativistic quantum field theories, though it is reassuring that in three-dimensional space-time the relativistic φ^4 interaction is known to be alive and well, just like the "improved" non-relativistic δ-function interaction.

The procedures for defining two- and three-dimensional δ-function interactions are two fold. One may simply perform *infinite* renormalization, arriving at amplitudes parametrized by a finite (by definition) *renormalized* coupling, which in the case of attraction may be alternatively expressed in terms of an uncalculable bound state energy. More satisfactory, especially within a mathematical frame, is the view that the δ-function interaction is merely a self-adjoint extension to a formally Hermitian, non-interacting Hamiltonian on a space with one point removed. The parameter in the extension is finite and plays the role of renormalized coupling strength.

These approaches to the δ-function interaction are as old as the subject. The first analyses, in three dimensions, were performed in physicists' terms by Bethe, Peierls and Fermi.[9] Mathematically rigorous treatments begin with the work of Berezin and Faddeev[10] and now there is even a monograph on the subject.[11] These days the two-dimensional δ-function interaction, and the equivalent self-adjoint extension have arisen in discussions of point particle dynamics in (2+1)-dimensional gravity[12] and in Chern–Simons gauge theory (Aharonov–Bohm/Ehrenberg-Siday interaction).[13]

Section II is devoted to qualitative remarks about quantum mechanical interactions and their symmetries. The δ-function Hamiltonians that we shall consider are introduced. The two-dimensional model is especially noteworthy, because on the classical/formal level it possesses a dilation symmetry, which is then *necessarily* destroyed by quantization — an effect seen in quantum field theory as the anomaly phenomenon,[14] but not previously identified in quantum mechanics.

In Section III the Bég–Furlong–Huang calculation is reconsidered but with infinite renormalization yielding non-trivial dynamics, and this is repeated in two dimensions. The same results are regained by the method of self-adjoint extension, whose symmetry breaking properties in two dimensions are highlighted.

In Section IV, the Aharonov–Bohm/Ehrenberg–Siday effect in the Dirac equation is shown to lead in an equivalent Schrödinger equation to a δ-function interaction, which for consistency *must* be interpreted as a self-adjoint extension. In this way the identification of the δ-function interaction with the self-adjoint extension is made complete.

The final Section V comprises concluding remarks.

II. DISCUSSION

The typical quantum mechanical Hamiltonian operator consists of a kinetic term involving spatial derivatives and a potential function of position. Of course, the expression appears Hermitian. However, appearances can be misleading, and experience shows that when the short-distance behavior of the potential function is the same or more singular than that of the kinetic operator pathologies can mar the eigenvalue problem. These arise because the Hamiltonian operator though formally Hermitian ("symmetric" in mathematical terminology) is not self-adjoint (the domain of definition of the operator does not coincide with the domain of the adjoint).

Some familiar examples: The non-relativistic r^{-2} potential shares an r^{-2} singularity with the Laplacian kinetic operator; when the potential is too strongly attractive (the strength depends on the dimension of space) the bound state spectrum is not discrete. For the Dirac–Coulomb problem, both the kinetic term $\boldsymbol{\alpha}\cdot\frac{1}{i}\nabla$ and the r^{-1} Coulomb potential behave as an inverse length at short distances, and the bound state energies become complex for sufficiently strong attraction.

Less familiar, but similar examples arise with vector potentials in Dirac theory: the Hamiltonian $\boldsymbol{\alpha}\cdot\left(\frac{1}{i}\nabla - \mathbf{A}\right)$, is not self-adjoint when $\mathbf{A} \propto r^{-1}$ at small distances, as for a point monopole in three dimensions[15] or a point vortex in two dimensions.[16]

The above information points to the following conclusions about Hamiltonians with δ-function potentials in non-relativistic Schrödinger equations for various dimensions.

$$H = \frac{1}{2}p^2 + v\delta(\mathbf{r}) = -\frac{1}{2}\nabla^2 + v\delta(\mathbf{r}) \tag{2.1}$$

(Mass and $\hbar$ are set to unity.) In two and three dimensions, where a δ-function scales as r^{-n}, $n = 2$ and 3, the short-distance singularity of the potential is respectively comparable to, and more singular than the kinetic term. Consequently, we anticipate difficulties with the eigenvalue equation.

$$H\psi = E\psi \tag{2.2}$$

Only the one-dimensional $\delta(x)$ potential presents a simple problem — one that is found in most quantum mechanical texts — but a $\delta'(x)$ potential exhibits pathologies similar to the above higher-dimensional cases.[11]

The two-dimensional δ-function [and the one-dimensional δ'-function] are additionally interesting in that the Hamiltonian does not contain dimensional parameters: v in (2.1) is dimensionless. This property, shared by the r^{-2} potential in any number of dimensions, renders the theory scale invariant, at least formally.

Specifically, what is meant here is that there exists a dilation operator D

$$D = tH - \frac{1}{4}\Big(\mathbf{r}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{r}\Big) \tag{2.3}$$

that implements the dilation transformation on the dynamical variable $\mathbf{r}$,

$$\delta_D \mathbf{r} = i\,[D, \mathbf{r}] = t\dot{\mathbf{r}} - \frac{1}{2}\mathbf{r} \tag{2.4}$$

showing that $\mathbf{r}$ has the scale dimension $-1/2$. Moreover, commuting with the Hamiltonian gives

$$i[D,H]=H \tag{2.5}$$

i.e. H has unit scale dimension. Therefore D is a constant of motion.

$$\frac{dD}{dt}=i[H,D]+\frac{\partial D}{\partial t}=0 \tag{2.6}$$

Establishing (2.5) requires the identity

$$i\left[\mathbf{r}\cdot\mathbf{p},\delta(\mathbf{r})\right]=\mathbf{r}\cdot\nabla\delta(\mathbf{r})=-2\delta(\mathbf{r}) \tag{2.7}$$

[In one dimension, what is needed is $i[xp,\delta'(x)]=x\delta''(x)=-2\delta'(x)$.] This insures that the interaction scales with $\mathbf{r}$ as r^{-2}, so that its scale dimension is unity.

One consequence of scale invariance is that quantum scattering phase shifts must be energy-independent (there is no scale to give an energy dependence!) — a fact which is explicitly verified by the energy-independent phase shifts of the r^{-2} potential.[16]

Another consequence, which follows from (2.3) in those simple models where $\mathbf{r}\cdot\mathbf{p}+\mathbf{p}\cdot\mathbf{r}=\frac{d}{dt}r^2$, is that D may be written as a total time derivative.

$$D=\frac{d}{dt}\left(\frac{t^2}{2}H-\frac{1}{4}r^2\right)$$

This reveals the presence of one more constant of motion,

$$K=-t^2H+2tD+\frac{1}{2}r^2 \tag{2.8}$$

$$\frac{dK}{dt}=0 \tag{2.9}$$

which is the conformal generator that transforms $\mathbf{r}$ according to

$$\delta_K\mathbf{r}=i[K,\mathbf{r}]=t^2\dot{\mathbf{r}}-t\mathbf{r} \tag{2.10}$$

Here we have another example of how scale invariance sometimes (but not always!) implies conformal invariance.

The operators H, K, D close on commutation: in addition to (2.5) it is true that

$$i[K,D]=K \tag{2.11}$$

$$i[H,K]=-2D \tag{2.12}$$

These commutators not only verify (2.9), but also show that the invariance algebra is $SO(2,1)$.

To conclude: $SO(2,1)$ is a symmetry group of the r^{-2} potential — a well-known fact that can be maintained quantum mechanically, provided there is not too much attraction[17] — and formally appears also to be a symmetry of the $\delta(\mathbf{r})$ potential in two dimensions. [$SO(2,1)$ is also known to be a quantum mechanical symmetry of the non-relativistic point magnetic monopole[18] and point vortex.[19]]

Our reason for entering upon this discussion of $SO(2,1)$ dynamical symmetry is that we shall soon establish the remarkable result that any quantum mechanical definition of the apparently $SO(2,1)$ invariant, two-dimensional $\delta(\mathbf{r})$ potential, necessarily violates $SO(2,1)$ invariance. Doubtlessly this is the most elementary manifestation of quantum mechanical symmetry breaking.

III. SOLVING THE δ-FUNCTION SCHRÖDINGER EQUATION

A. Renormalization

We seek scattering solutions to (2.1) and (2.2) for the spatial dimensionalities n, $n=2,3$, which we treat simultaneously.

In terms of momentum-space wave functions

$$\varphi(\mathbf{p})=\int d^n\mathbf{r}\,e^{i\mathbf{p}\cdot\mathbf{r}}\psi(\mathbf{r}) \tag{3.1}$$

equations that we solve are

$$\begin{aligned}\frac{1}{2}\left(p^2-k^2\right)\varphi(\mathbf{p})&=-v\psi(\mathbf{0})\\ \frac{k^2}{2}&=E\end{aligned} \tag{3.2}$$

The scattering solutions are

$$\varphi(\mathbf{p})=(2\pi)^n\delta(\mathbf{p}-\mathbf{k})-\frac{2v}{p^2-k^2-i\epsilon}\psi(\mathbf{0}) \tag{3.3}$$

Evidently the scattering amplitudes are proportional to $v\psi(\mathbf{0})$, which is self-consistently determined from (3.3) by

$$\begin{aligned}\psi(\mathbf{0})&=\int\frac{d^n\mathbf{p}}{(2\pi)^n}\varphi(\mathbf{p})=1-2vI_n\left(-k^2-i\epsilon\right)\psi(\mathbf{0})\\ v\psi(\mathbf{0})&=\left(\frac{1}{v}+2I_n\left(-k^2-i\epsilon\right)\right)^{-1}\end{aligned} \tag{3.4}$$

where I_n is the integral

$$I_n(z)=\int\frac{d^n\mathbf{p}}{(2\pi)^n}\,\frac{1}{p^2+z} \tag{3.5}$$

which diverges in the ultraviolet for $n\geq 2$.

To make progress, we regulate, for example by limiting $|\mathbf{p}|$ at Λ. It follows for large Λ that

$$I_2^\Lambda(z)=\frac{1}{4\pi}\ln\frac{\Lambda^2}{z} \tag{3.6}$$

$$I_3^\Lambda(z)=\frac{1}{2\pi^2}\Lambda-\frac{1}{4\pi}\sqrt{z} \tag{3.7}$$

Alternatively one may use dimensional regularization and find

$$I_2^\epsilon(z)=\frac{1}{4\pi}\ln\frac{4\pi\,e^{-\gamma+1/\epsilon}}{z} \tag{3.8}$$

$$I_3^\epsilon(z)=-\frac{1}{4\pi}\sqrt{z} \tag{3.9}$$

In (3.8) we calculate with $n = 2 - 2\epsilon$ and take the limit $\epsilon \to 0$, obtaining a result identical to (3.6). In (3.9) there is no dependence on ϵ, the departure from three dimensions; a finite answer is obtained as is characteristic for dimensionally regulated, odd-dimensional integrals. From (3.4) $v\psi(0)$ is determined for the two cases as

$$n = 2: \quad v\psi(0) = \left(\frac{1}{v} + \frac{1}{\pi}\ln\frac{\Lambda}{\mu} - \frac{1}{\pi}\ln\frac{k}{\mu} + \frac{i}{2}\right)^{-1} \tag{3.10}$$

$$n = 3: \quad v\psi(0) = \left(\frac{1}{v} + \frac{1}{\pi^2}\Lambda + \frac{ik}{2\pi}\right)^{-1} \tag{3.11}$$

except that with dimensional regularization $\frac{1}{\pi^2}\Lambda$ is absent from (3.11). In (3.10) μ is a convenient normalization point.

As Λ is removed to infinity, $v\psi(0)$ and therefore the scattering amplitudes vanish, both for $n = 2$ and $n = 3$, *provided v is finite*. At $n = 3$ the result of Bég–Furlong and Huang is thus regained, but note the curiosity that with dimensional regularization the three-dimensional answer is cut-off independent and finite. Moreover, the two-dimensional scattering amplitude vanishes for both regularization procedures.

In the spirit of quantum field theory, it is very plausible to take the bare coupling to be cut-off dependent and to introduce a renormalized coupling constant g, in terms of which (3.10) and (3.11) read

$$n = 2: \quad v\psi(0) = \left(\frac{1}{g} - \frac{1}{\pi}\ln\frac{k}{\mu} + \frac{i}{2}\right)^{-1} \tag{3.12}$$

$$n = 3: \quad v\psi(0) = \left(\frac{1}{g} + \frac{ik}{2\pi}\right)^{-1} \tag{3.13}$$

These are finite and well-defined, provided $1/v$ absorbs by definition the cut-off dependence.

$$n = 2: \quad \frac{1}{g} = \frac{1}{v} + \frac{1}{\pi}\ln\frac{\Lambda}{\mu} \tag{3.14}$$

$$n = 3: \quad \frac{1}{g} = \frac{1}{v} + \frac{1}{\pi^2}\Lambda \tag{3.15}$$

To obtain the scattering amplitude, we present (3.3) in position space

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - 2vG_k(r)\psi(0) \tag{3.16}$$

where $G_k(r)$ are the Green's functions appropriate to the two dimensionalities.

$$\left(-\nabla^2 - k^2\right)G_k(r) = \delta(\mathbf{r}) \tag{3.17}$$

$$n = 2: \quad G_k(r) = \frac{i}{4}H_0^{(1)}(kr) \underset{r\to\infty}{\longrightarrow} \frac{1}{2\sqrt{2\pi kr}}e^{i(kr+\pi/4)} \tag{3.18}$$

$$n = 3: \quad G_k(r) = \frac{1}{4\pi}\frac{e^{ikr}}{r} \tag{3.19}$$

Upon identifying the scattering amplitude from the asymptotic behavior of the scattering wave function,

$$n = 2: \quad \psi(\mathbf{r}) \longrightarrow e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{\sqrt{r}}f(\theta)e^{i(kr+\pi/4)} \tag{3.20}$$

$$n = 3: \quad \psi(\mathbf{r}) \longrightarrow e^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{r}f(\theta)e^{ikr} \tag{3.21}$$

we obtain

$$n = 2: \quad f(\theta) = -\frac{1}{\sqrt{2\pi k}}v\psi(0) = -\frac{1}{\sqrt{2\pi k}}\left(\frac{1}{g} - \frac{1}{\pi}\ln\frac{k}{\mu} + \frac{i}{2}\right)^{-1} \tag{3.22}$$

$$n = 3: \quad f(\theta) = -\frac{1}{2\pi}v\psi(0) = -\left(\frac{2\pi}{g} + ik\right)^{-1} \tag{3.23}$$

Only s-wave scattering takes place, whose phase shifts may be read off from standard formulas.

$$n = 2: \quad f(\theta) = \frac{1}{i\sqrt{2\pi k}}\sum_{m=-\infty}^{\infty}\left(e^{2i\delta_m} - 1\right)e^{im\theta} \tag{3.24}$$

$$n = 3: \quad f(\theta) = \frac{1}{2ik}\sum_{\ell=0}^{\infty}\left(e^{2i\delta_\ell} - 1\right)P_\ell(\cos\theta) \tag{3.25}$$

Comparison with (3.22) and (3.23) yields

$$n = 2: \quad \operatorname{ctn}\delta_0 = \frac{1}{\pi}\ln\frac{k^2}{\mu^2} - \frac{2}{g} \tag{3.26}$$

$$n = 3: \quad \tan\delta_0 = -\frac{gk}{2\pi} \tag{3.27}$$

Note that the two-dimensional scattering phase shift has acquired an $\ln k$ dependence, in clear violation of scale invariance, except of course at $g = 0$, which corresponds to no interaction.

With attractive δ-functions there exist bound states, provided v is renormalized. Then in the scattering amplitudes the renormalized coupling constant may be replaced by the bound state energy, which in two dimensions is an example of dimensional transmutation within quantum mechanics.[20]

The bound state, momentum space wave functions $\varphi_B(\mathbf{p})$ satisfy (3.2) with $E=-B$, and the solution are

$$\varphi_B(\mathbf{p}) = -\frac{2v\psi_B(0)}{p^2+2B} \tag{3.28}$$

This implies

$$\psi_B(0) = -2v\int \frac{d^n\mathbf{p}}{(2\pi)^n}\,\frac{1}{p^2+2B}\psi_B(0) \tag{3.29a}$$

or

$$-\frac{1}{2v} = I_n(2B) \tag{3.29b}$$

which should be used to determine B in terms of v, but of course (3.29) exhibits divergences that shall be discussed presently. First let us express everything in terms of the bound state energies B.

The wave function normalizations fix $2v\psi_B(\mathbf{0})$.

$$n=2: \quad \varphi_B(\mathbf{p}) = \sqrt{8\pi B}\frac{1}{p^2+2B} \tag{3.30}$$

$$n=3: \quad \varphi_B(\mathbf{p}) = \left(128\pi^2 B\right)^{1/4}\frac{1}{p^2+2B} \tag{3.31}$$

Equation (3.29b) may be used to eliminate $1/v$ in the scattering solutions. In (3.4) $v\psi(\mathbf{0})$ becomes

$$v\psi(\mathbf{0}) = \frac{1}{2}\left[I_n\left(-k^2-i\epsilon\right) - I_n(2B)\right]^{-1} \tag{3.32}$$

Since a single subtraction renders I_n finite for $n=2$ and 3, the scattering amplitudes can be expressed in terms of "physical" quantities.

$$n=2: \quad f(\theta) = -\frac{1}{\sqrt{2\pi k}}\left(\frac{1}{2\pi}\ln\frac{2B}{k^2}+\frac{i}{2}\right)^{-1} \tag{3.33}$$

$$n=3: \quad f(\theta) = -\left(\sqrt{2B}+ik\right)^{-1} \tag{3.34}$$

Because of the divergences, bound state energies are not calculable: from (3.6), (3.7) [or (3.8), 3.9)] and (3.29b) we get

$$n=2: \quad -\frac{1}{v} = \frac{1}{2\pi}\ln\frac{\Lambda^2}{2B} \tag{3.35}$$

$$n=3: \quad -\frac{1}{v} = \frac{1}{\pi^2}\Lambda - \frac{1}{2\pi}\sqrt{2B} \tag{3.36}$$

The Λ dependence may be combined with $1/v$ and hidden in the renormalized coupling $1/g$ as in (3.14) and (3.15). Then B is related to g by

$$n=2: \quad \sqrt{2B} = \mu e^{\pi/g} \tag{3.37}$$

$$n=3: \quad \sqrt{2B} = \frac{2\pi}{g} \tag{3.38}$$

This is also seen in (3.22), (3.23): the scattering amplitudes posses poles on the positive imaginary k axis corresponding to the above bound state energies. Note that (3.38) requires g to be positive, but no sign restriction on g need be made in (3.37).

Formulas (3.35) and (3.36) may be used to put into evidence a physical effect. A regulated expression for the δ-function potential is posited

$$\begin{gathered} V_n(\mathbf{r}) = \frac{c_n v}{2\pi R^{n-1}}\delta(r-R) \\ c_2 = 1\ , \qquad c_3 = \frac{1}{2} \end{gathered} \tag{3.39}$$

that effectively reproduces the potential in (2.1) when $R\to 0$. V_n supports s-wave bound states. The binding energies B are obtained by matching the discontinuities at $r=R$ in the logarithmic derivatives of the wave functions against the coefficients of the δ-function (3.39). This leads to the equations

$$n=2: \quad -\frac{1}{v} = \frac{1}{\pi}I_0\left(\sqrt{2B}\,R\right)K_0\left(\sqrt{2B}\,R\right) \approx \frac{1}{2\pi}\ln\frac{4e^{-2\gamma}/R^2}{2B} \tag{3.40}$$

$$n=3: \quad -\frac{1}{v} = \frac{1}{2\pi R}\,\frac{1-e^{-\sqrt{8B}\,R}}{\sqrt{8B}\,R} \approx \frac{1}{2\pi R} - \frac{1}{2\pi}\sqrt{2B}\ . \tag{3.41}$$

The second approximate equalities are valid as the regulator is removed and R is small; they are seen to reproduce (3.35) and (3.36) with $\Lambda \propto R^{-1}$. Similar results are obtained by a square well or lattice regularization of the δ-function.[21]

B. Self-Adjoint Extension

Although regularization and infinite renormalization of the δ-function interaction strength produces physically sensible (unitary) and non-trivial scattering amplitudes, and also the possibility of bound states, it would seen preferable to arrive at the results without introducing the mathematically awkward "infinite" quantity $\Lambda \propto R^{-1}$. This can be achieved through the method of self-adjoint extension.[11]

Consider the free Schrödinger operator.

$$H^0 = \frac{1}{2}p^2 = -\frac{1}{2}\nabla^2 \tag{3.42}$$

This is Hermitian and self-adjoint when acting on functions that are finite. However, for two and three dimensions H^0 remains self-adjoint even when the finiteness requirement is relaxed: functions are permitted to diverge at isolated points, provided they remain square integrable; and also a boundary condition, consistent with self-adjointness of H^0, needs to be specified. We take a single point of divergence, the origin, and the required boundary condition is imposed on s-waves; it involves an arbitrary parameter, the self-adjoint extension parameter λ.[11]

$$n=2: \quad \lim_{r\downarrow 0}\frac{\psi(r)}{\ln r}=\frac{\lambda}{\pi}\lim_{r\downarrow 0}\left(\psi(r)-\lim_{r'\downarrow 0}\frac{\psi(r')}{\ln r'}\ln r\right) \tag{3.43}$$

$$n=3: \quad \lim_{r\downarrow 0} r\psi(r)=-\frac{\lambda}{2\pi}\lim_{r\downarrow 0}\left(\psi(r)+r\psi'(r)\right) \tag{3.44}$$

This defines the extended Hamiltonians H^λ; $\lambda=0$ corresponds to the conventional free Hamiltonian with regular wave functions.

It is obvious that with (3.43) and (3.44) one can find s-wave eigenfunctions of H^λ that differ from the regular non-interacting ones $J_0(kr)$ (two dimensions) and $\sin kr/r$ (three dimensions), because the irregular solution is now acceptable. For positive energy $E=k^2/2$ we have

$$n=2: \quad \psi(r)=J_0(kr)-\tan\delta Y_0(kr) \tag{3.45}$$

$$n=3: \quad \psi(r)=\frac{1}{r}\left(\sin kr+\tan\delta\cos kr\right) \tag{3.46}$$

From (3.43) one determines that δ in (3.45) coincides with the phase shift δ_0 of (3.26) when $1/g$ is identifies with $1/\lambda-\gamma/\pi-(1/\pi)\ln\mu/2$, while with (3.44) δ in (3.46) is the same as δ_0 in (3.27), with $\lambda=g$. Also there are bound states, $E=-B$.

$$n=2: \quad \psi_B(r)=\sqrt{\frac{2B}{\pi}}K_0\left(\sqrt{2B}\,r\right) \tag{3.47}$$

$$n=3: \quad \psi_B(r)=\left(\frac{B}{2\pi^2}\right)^{1/4}\frac{e^{-\sqrt{2B}\,r}}{r} \tag{3.48}$$

The binding energies, fixed by satisfying (3.43) and (3.44), agree with (3.37) and (3.38), once the above identifications between g and λ are made. As before the sign of $\lambda(g)$ is immaterial for the two-dimensional bound state, while for the three-dimensional bound state, it must be positive. Finally, it is readily verified that (3.47), (3.48) are the Fourier transforms of (3.30), (3.31).

In conclusion we see that the method of self-adjoint extension provides a description of renormalized δ-function potentials in two and three spatial dimensions for the following three reasons:

1) It is *a priori* plausible to describe a Hamiltonian with a δ-function potential as a free Hamiltonian on a space with one point deleted plus a boundary condition specifying what happens at that point.

2) The boundary conditions (3.43) and (3.44) permit a $\ln r$ and a r^{-1} singularity in the two- and three-dimensional wave functions., as is seen in (3.45), (3.47) and (3.46), (3.48). The effect of the Laplacian on these is indeed a δ-function.
3) Most convincing is the fact that the scattering data and bound state spectra arising from the renormalized δ-function interactions are reproduced by the self-adjoint extensions.

In the next Section, another reason for viewing a δ-function potential as a self-adjoint extension is given.

The boundary condition (3.43) implied by the self-adjoint extension in two dimensions shows also why dilation symmetry is broken quantum mechanically. Observe first that the logarithms occurring in that equation introduce a scale for r, which is not dilation invariant. More formally, we can demonstrate that D is not defined on our space. Consider any s-wave energy eigenfunction $\psi_E(r)$. From (2.3) it follows that

$$D\psi_E(r)=tE\psi_E(r)+\frac{i}{2}\left(r\partial_r+1\right)\psi_E(r) \tag{3.49}$$

The boundary condition (3.43) requires that at small r, $\psi_E(r)$ is proportional to $1+\frac{\lambda}{\pi}\ln r$. But then the last term in (3.49) is proportional to $1+\frac{\lambda}{\pi}+\frac{\lambda}{\pi}\ln r$, which means that $D\psi_E$ satisfies (3.43) and exists in the Hilbert space only for $\lambda=0$.[22]

IV. AHARONOV–BOHM / EHRENBERG–SIDAY INTERACTION

The two-dimensional δ-function potential arises also in a problem involving a point magnetic vortex. The interpretation as a self-adjoint extension serves to explain a puzzle that occurs in this context.

Let us first describe the problem and the puzzle. Consider the Hamiltonian for a planar Dirac particle interacting with a magnetic field B, described by the vector potential $\mathbf{A}$, $B=\epsilon^{ij}\partial_i A^j$.

$$H=\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{A})+\beta m \tag{4.1}$$

In $(2+1)$ space-time dimensions, Dirac matrices are 2×2 and may be chosen to be the Pauli matrices: $\alpha^i=\sigma^i$, $i=1,2$; $\beta=\sigma^3$.

The Dirac eigenvalue equation for the two-component spinor $\chi=\binom{\chi_+}{\chi_-}$

$$\left(\boldsymbol{\alpha}\cdot(\mathbf{p}-\mathbf{A})+\beta m\right)\chi=\epsilon\chi \tag{4.2}$$

may be iterated and decoupled.

$$\left((\mathbf{p}-\mathbf{A})^2-\beta B\right)\chi=\left(\epsilon^2-m^2\right)\chi \tag{4.3a}$$

Thus we arrive at two Schrödinger equations for the two components $\chi_\pm$.

$$\left(\frac{1}{2}\left(\frac{1}{i}\nabla - \mathbf{A}\right)^2 \mp \frac{1}{2}B\right)\chi_\pm = E\chi_\pm$$
$$E = \frac{1}{2}(\epsilon^2 - m^2) \tag{4.3b}$$

Now for the puzzle: we consider a point vortex, as in an idealized description of the Aharonov–Bohm/Ehrenberg–Siday effect.

$$B = \Phi\delta(\mathbf{r})$$
$$A^i = -\frac{\Phi}{2\pi}\epsilon^{ij}\frac{r^j}{r^2} = \frac{\Phi}{2\pi}\partial_i\theta \tag{4.4}$$

Since the magnetic field vanishes almost everywhere, the vector potential is a pure gauge almost everywhere — it is expressed in (4.4) as a gradient of the angle θ, $\tan\theta = \frac{y}{x}$, $r = (x, y)$. [Consistency requires the amusing formula $\epsilon^{ij}\partial_i\partial_j\theta = 2\pi\delta(\mathbf{r})$.] Since $\mathbf{A}$ is a pure gauge it may be removed from the equations by defining

$$\chi = e^{i\nu\theta}\chi^0 \tag{4.5}$$

where $\nu = \Phi/2\pi$. However, since χ is single-valued,

$$\chi\big|_{\theta=2\pi} = \chi\big|_{\theta=0} \tag{4.6a}$$

it follows that χ^0 satisfies

$$\chi^0\big|_{\theta=2\pi} = e^{-2\pi\nu i}\chi^0\big|_{\theta=0} \tag{4.6b}$$

So χ^0 is "multivalued."

When the change of variables (4.5) is made in the Dirac equation (4.2) we find that χ^0 satisfies the free Dirac equation,

$$(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\,\chi^0 = \epsilon\chi^0 \tag{4.7}$$

and the interaction is entirely hidden in the angular boundary condition (4.6b). On the other hand, changing to χ^0 in the Schrödinger equation (4.3), removes the vector potential but leaves the magnetic field, which is here a δ-function,

$$\left(-\frac{1}{2}\nabla^2 \mp \pi\nu\delta(\mathbf{r})\right)\chi^0_\pm = E\chi^0_\pm \tag{4.8}$$

while the boundary condition, on $\chi^0_\pm$ remain as in (4.6b). The question now presents itself: does χ^0 satisfy a free equation as in (4.7) or does it experience an additional δ-function interaction as in (4.8)? It appears puzzling that both equations are true.

The answer to the question and the resolution to the puzzle resides in the self-adjoint extension that has to be performed on the Dirac equation. No matter whether the equation is taken with its interaction as in (4.2), or without an interaction as in (4.7), but with the angular boundary condition (4.6b), it is impossible to satisfy it for $\nu \neq 0$ with wave functions that are everywhere finite. One must admit an infinite but normalizable solution and once this is allowed, a further boundary condition must be specified. In other words neither Dirac operator is self-adjoint and a self-adjoint extension is required.[13] On the other hand, in view of what was explained in Section III, the Schrödinger equation (4.8) with a δ-function should be viewed as the free equation with self-adjoint extension. Therefore regardless whether one works with (4.7) or (4.8), the Hamiltonian is non-interacting, there is an angular boundary condition (4.6b) that recalls the presence of the vortex, and there is further radial boundary condition specifying the self-adjoint extension.

It must be emphasized that an important difference exists between the self-adjoint extensions of the free Schrödinger Hamiltonian $H^0_S \equiv -\frac{1}{2}\nabla^2$ with conventional angular boundary conditions and of the free Dirac Hamiltonian $H^0_D \equiv \boldsymbol{\alpha}\cdot\frac{1}{i}\nabla + \beta m$ with vortex angular boundary conditions. In the former, no extension is needed; H^0_S has finite eigenfunctions and an extension represents an additional interaction — the δ-function. On the other hand, the free vortex Dirac Hamiltonian H^0_D does not possess finite eigenfunctions; an extension is required and it represents further information (beyond total flux) that must be specified when describing the physical attributes of the already posited vortex. In contrast to the Schrödinger case, the extension in the Dirac equation is not a matter of choice and does not reflect *additional* interactions.[23]

V. CONCLUSION

There can be no doubt that a δ-function interaction in two- and three-dimensional Schrödinger theory can be defined, and the method of self-adjoint extension allows dispensing with infinite renormalization. Of course the relation of these non-relativistic theories to relativistic field theories in $(2+1)$- and $(3+1)$-dimensional space-time is purely formal. Thus one cannot draw any definite conclusions about the field theory models.

In $(1+1)$-dimensional space-time, the φ^4 relativistic interaction rigorously goes over to the non-relativistic Schrödinger theory of a one-dimensional δ-function.[24] Of course, the possibility of carrying out the proof relies on the mildness of that field theory's ultraviolet divergences. It should also be feasible to carry out an analysis of the non-relativistic limit for the super-renormalizable $(2+1)$-dimensional φ^4 theory. While this model is known to exist, the presumed Schrödinger theory δ-function limit shows some unexpected features: the need for infinite renormalization, which is not necessary in the field theory; the existence of a bound state, regardless of the sign of the renormalized coupling — it is as if only an attractive non-relativistic theory exists.

In conclusion, while the status of relativistic, $(3+1)$-dimensional φ^4 field theory remains unsettled, the non-relativistic theory is not necessarily trivial. Indeed a non-trivial scattering amplitude exists, but its construction is a subtle task. It remains to be seen whether a subtle construction of the field theory is possible.

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