Mathematical Preliminaries + Notation

---

6 Postulates of QM

some specific time
not a variable

① At each instant of time (say t_0) the <u>state</u> of a physical system is represented by a <u>ket</u> $|\psi\rangle$ in the abstract mathematical space of states, a Hilbert space.

related to but not the same as the wavefunction $\psi(x)$

- Dirac Notation: ket $|\psi\rangle$

  bra $\langle\psi|$ → bra c ket

- The space is <u>not</u> 3-dimensional x, y, z coordinate space.

- (Linear) <u>Vector Space</u>

$$|\Psi\rangle = \alpha|\Psi_1\rangle + \beta|\Psi_2\rangle$$

complex numbers
c-number

$\Rightarrow$ superposition

Space could by infinite dimsional.

- There exists a <u>norm</u>,

a "distance" positive definite

$\Rightarrow$ Banach space

$$\text{norm } |\Psi\rangle = \langle\Psi|\Psi\rangle^{1/2} \geq 0$$

$$\langle\Psi|\Psi\rangle^* = \langle\Psi|\Psi\rangle \text{ real}$$

- The norm is an inner product (dot product)

$\Rightarrow$ Hilbert space.

$$\langle \psi | \varphi \rangle = \int_{\text{All}} dx\, \psi^*_{(x)}\, \varphi_{(x)}$$

x-basis

wavefunctions

Parseval's theorem

$$= \int_{\text{All}} dp\, \tilde{\psi}^*_{(p)}\, \tilde{\varphi}_{(p)}$$

momentum p-basis

Plancherel's Theorem

$\tilde{\psi}_{(p)}$ is the Fourier transform of $\psi_{(x)}$

$$\langle x | \varphi \rangle \equiv \varphi_{(x)}$$

Inner Product: $\langle \psi | \varphi \rangle^* = \langle \varphi | \psi \rangle$

complex numbers (complex conjugates)

② Every observable of a system corresponds to an operator that acts on kets: $\hat{A}|\psi\rangle$

In general $\hat{A}|\psi\rangle = |\varphi\rangle$ ← different ket

$|\varphi\rangle \neq \alpha|\psi\rangle$

but ∃ special kets "eigenstates" such that $\hat{A}|\chi_j\rangle = a_j|\chi_j\rangle$

operator ket c-number same ket

eigenvalue

③ The only possible result of a measurement of an observable is one of its eigenvalues.

- measurements are real

$$\Rightarrow \hat{A}^{\dagger} = \hat{A} \quad \text{hermitian}$$

transpose, complex conjugate (dagger)

- eigenstates are orthogonal

$$\langle \chi_i | \chi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

i, j discrete
1, 2, 3, ...

Kronecker delta

---

i, j continuous

$$\langle \chi(i) | \chi(j) \rangle = \delta(i-j)$$

$$\langle \chi(y) | \chi(z) \rangle = \delta(y-z) \qquad y, z \text{ real}$$

• eigenkets span the Hilbert space and form a basis.

Can expand any ket

$$|\psi\rangle = \sum_i c_i |\chi_i\rangle$$

integral if continuous

Think of $|\psi\rangle$ like $\vec{v}$

$$\vec{v} = v_x \hat{\underline{x}} + v_y \hat{\underline{y}} + v_z \hat{\underline{z}}$$ Cartesian basis

or $$\vec{v} = v_r \hat{r} + v_\theta \hat{\theta} + v_\varphi \hat{\varphi}$$ Spherical Polar basis