④ When a measurement of an observable $\hat{A}$ is made on a generic state $|\psi\rangle$, the probability of obtaining eigenvalue a_n is $|\langle \chi_n|\psi\rangle|^2$ where $\hat{A}|\chi_n\rangle = a_n|\chi_n\rangle$

- Normalized $\langle\psi|\psi\rangle = 1$, $\langle \chi_j|\chi_n\rangle = \delta_{jn}$
- $\langle \chi_n|\psi\rangle$ is a complex number called the "probability amplitude". (square root of probability.)
- $|\psi\rangle = \sum_n c_n |\chi_n\rangle$ left multiply by $\langle \chi_j|$

$$\langle \chi_j|\psi\rangle = \sum_n c_n \langle \chi_j|\chi_n\rangle = \sum_n c_n \delta_{jn} = c_j$$

change $\boxed{c_n = \langle \chi_n|\psi\rangle}$ substitute

$$|\psi\rangle = \sum_n \left(\langle \chi_n|\psi\rangle\right)|\chi_n\rangle = \underbrace{\left[\sum_n |\chi_n\rangle\langle \chi_n|\right]}_{\hat{\mathbb{1}} \text{ identity operator}}|\psi\rangle$$

Completeness: $\sum_n |\chi_n\rangle\langle \chi_n| = \hat{\mathbb{1}}$

• The probabity of getting <u>some</u> result is unity

$$1 = |\langle\Psi|\Psi\rangle|^2 = \sum_m \sum_n c_m^* c_n \underbrace{\langle \chi_m|\chi_n\rangle}_{\delta_{mn}}$$

$$= \boxed{\sum_n |c_n|^2 = 1} \qquad |c_n|^2 = c_n^* c_n \ \text{Real}$$

• Expectation value of $\hat{A}$ in the state $|\Psi\rangle$

$$\langle \hat{A}\rangle_\Psi \equiv \langle\Psi|\hat{A}|\Psi\rangle = \left(\sum_m c_m^* \langle\chi_m|\right)\hat{A}\left(\sum_n c_n|\chi_n\rangle\right)$$

$$= \sum_m \sum_n c_m^* c_n a_n \underbrace{\langle\chi_m|\chi_n\rangle}_{\delta_{mn}} = \sum_n |c_n|^2 a_n$$

5 Immediately after measurement of $\hat{A}$ has yielded an eigenvalue a_n, the state of the system is the normalized eigenstate $|\chi_n\rangle$. not $|c_n|^2|\chi_n\rangle$

"Collapse of the wavefunction" violates unitarity.

• not deterministic, but probabilistic.

solution is called decoherence.

⑥ The time evolution of a quantum system preserves the normalization of the ket $|\psi\rangle$.

deterministic

$$|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle \qquad \hat{U}^{\dagger} = \hat{U}^{-1}$$

where $\hat{U}$ is a unitary operator : $\hat{U}^{\dagger}\hat{U} = \hat{\mathbb{1}}$

$\hat{U}^{\dagger}$ hermitian conjugate, dagger, adjoint

This postulate implies the Schrödinger Equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

where $\hat{H}$ is a hermitian op. $(\hat{H}^{\dagger} = \hat{H})$

- $|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle$ evolve deterministically forward or backward in time.

- unitarity $\hat{U}^{\dagger}(t,t_0)\,\hat{U}(t,t_0) = \hat{\mathbb{1}}$

- norm of state is preserved

$$\langle\psi(t)|\psi(t)\rangle = \left(\langle\psi(t_0)|\underbrace{\hat{U}^{\dagger}(t,t_0)\right)\left(\hat{U}(t,t_0)}_{\mathbb{1}}|\psi(t_0)\rangle\right)$$

$$= \langle\psi(t_0)|\psi(t_0)\rangle$$

- idenity $t=t_0$ no time evolution

$$\hat{U}(t_0,t_0)=\hat{\mathbb{1}} \quad \forall\, t_0$$

- composition

$$|\psi(t_2)\rangle = \hat{U}(t_2,t_1)|\psi(t_1)\rangle = \hat{U}(t_2,t_1)\hat{U}(t_1,t_0)|\psi(t_0)\rangle$$
$$= \hat{U}(t_2,t_0)|\psi(t_0)\rangle$$

$$\Rightarrow \hat{U}(t_2,t_0) = \hat{U}(t_2,t_1)\hat{U}(t_1,t_0)$$

- inverse $\hat{U}(t_0,t)\hat{U}(t,t_0) = \hat{\mathbb{1}}$

$$\Rightarrow \hat{U}(t_0,t) = \left[\hat{U}(t,t_0)\right]^{-1} = \hat{U}^{\dagger}(t,t_0)$$

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]|\psi(t_0)\rangle$$

independent of t.

$$= \left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]\hat{U}(t_0,t)|\psi(t)\rangle$$

$$= \underbrace{\left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]\hat{U}^{\dagger}(t,t_0)}_{\equiv \hat{A}(t,t_0)}|\psi(t)\rangle$$

→ $\hat{A}(t,t_0)$ is antihermitian

$$\hat{U}(t,t_0)\hat{U}^\dagger(t,t_0) = \hat{\mathbb{1}}$$

time derivative both sides

$$\underbrace{\left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]\hat{U}^\dagger(t,t_0)}_{\hat{A}(t,t_0)} + \underbrace{\hat{U}(t,t_0)\frac{\partial U^\dagger(t,t_0)}{\partial t}}_{\hat{A}^\dagger(t,t_0)} = 0$$

$$\hat{A}(t,t_0) + \hat{A}^\dagger(t,t_0) = 0$$

$$\Rightarrow \hat{A}(t,t_0) = -\hat{A}^\dagger(t,t_0) \quad ✓$$

→ $A(t,t_0)$ is independent of t_0.

$$\hat{A}(t,t_0) = \left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]\hat{U}^\dagger(t,t_0)$$

insert $\hat{\mathbb{1}} = \hat{U}(t_0,t_1)\hat{U}^\dagger(t_0,t_1)$

$$= \left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\hat{U}(t_0,t_1)\right]\underbrace{\underbrace{\hat{U}^\dagger(t_0,t_1)}_{\hat{U}(t_1,t_0)}\underbrace{\hat{U}^\dagger(t,t_0)}_{\hat{U}(t_0,t)}}_{\hat{U}(t_1,t)}$$

$$= \left[\frac{\partial \hat{U}(t,t_1)}{\partial t}\right]\hat{U}^\dagger(t,t_1)$$

$$= \hat{A}(t,t_1) \equiv \hat{A}(t) \quad ✓$$

$$\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{A}(t)|\Psi(t)\rangle$$

$\hat{A}$ is antihermitian

~~units~~ dimension of $\frac{1}{\text{time}}$

Define $\hat{H}(t) \equiv i\hbar\hat{A}(t) = i\hbar\left[\frac{\partial \hat{U}(t,t_0)}{\partial t}\right]\hat{U}^{\dagger}(t,t_0)$

hermitian, ~~units~~ dimension of energy.

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

Schrödinger equation

why is $\hat{H}$ the Hamiltonian (energy)