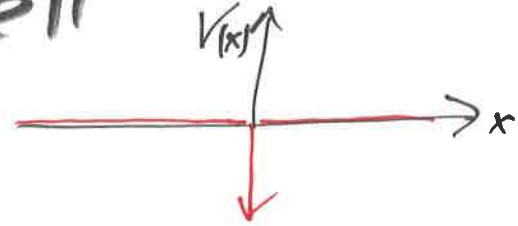
Delta function Well



$V(x) = -\lambda\,\delta(x)\,, \quad \lambda > 0$

If $E > 0$, all real E allowed, not quantized
non-denumerably many scattering states

Exactly one bound state $E < 0$

TISE: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \lambda\,\delta(x)\,\psi(x) = E\,\psi(x) \quad \forall x$

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for $x < 0$, $V(x) = 0$: $-\frac{\hbar^2}{2m}\psi_L''(x) = E\,\psi_L(x)$

2nd-order, linear in ψ, homogeneous

$\psi_L(x) = A e^{-\kappa x} + B e^{+\kappa x}\,, \quad \kappa \equiv \sqrt{\frac{-2mE}{\hbar^2}}$ real

$e^{-\kappa x} \to \infty$ as $x \to -\infty \implies A = 0$

$\underline{x < 0}$: $\psi_L = B e^{+\kappa x}$

for $x > 0$, again $V(x) = 0$ $\quad -\frac{\hbar^2}{2m}\psi_R''(x) = E\,\psi_R(x)$

$\psi_R(x) = C e^{-\kappa x} + D e^{+\kappa x}$

$e^{\kappa x} \to \infty$ as $x \to \infty$

$\implies D = 0$

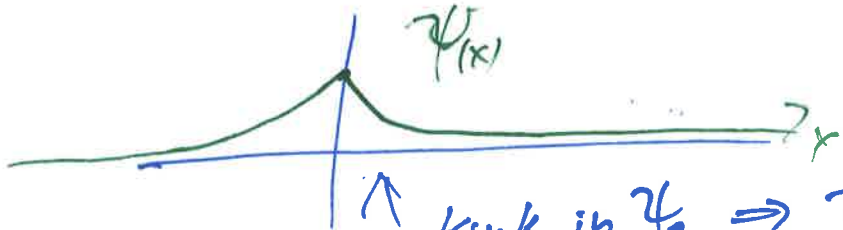
$\psi_R(x) = +C e^{-\kappa x}$

① $\psi(x)$ must be continuous

$$\psi_L(0) = \psi_R(0) \Rightarrow Be^{\kappa 0} = Ce^{-\kappa 0} \Rightarrow B = C$$

$$\psi(x) = \begin{cases} Be^{\kappa x}, & x \le 0 \\ Be^{-\kappa x}, & x \ge 0 \end{cases} \Rightarrow \psi(x) = Be^{-\kappa |x|}$$

$$= Be^{-\sqrt{\frac{-2mE}{\hbar^2}}|x|}$$



↑ kink in ψ_0 $\Rightarrow$ ψ' will be discontinuous at $x=0$.
"jump discontinuity"

Integrate the SE from $-\epsilon$ to $+\epsilon$, then limit $\epsilon \to 0$.

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}dx + \int_{-\epsilon}^{+\epsilon} V(x)\psi(x)dx = E\int_{-\epsilon}^{+\epsilon}\psi(x)dx$$

$\downarrow$ $\qquad\qquad\downarrow\qquad\qquad\downarrow$ in limit

$$-\frac{\hbar^2}{2m}\frac{d\psi}{dx}\Big|_{-\epsilon}^{+\epsilon} \quad -\lambda\psi(0) \quad = \quad 0$$

$$\psi'(\epsilon) - \psi'(-\epsilon) = \frac{-2m\lambda}{\hbar^2}\psi(0) \qquad \text{Limit } \epsilon \to 0$$

$$\psi_R'(0) - \psi_L'(0) = \frac{-2m\lambda}{\hbar^2}\psi(0)$$

↖ R or L

$$\psi_R'(x) = \frac{d}{dx} B e^{-Kx} = -BK e^{-Kx}; \quad \psi_R'(0) = -BK$$

$$\psi_L'(x) = \frac{d}{dx} B e^{+Kx} = +BK e^{Kx}, \quad \psi_L'(0) = BK$$

$$\psi(0) = B$$

$$-2\cancel{B}K = \frac{-2m\lambda}{\hbar^2}\cancel{B} \implies K = \frac{m\lambda}{\hbar^2}$$

$$E = -\frac{\hbar^2 K^2}{2m} = -\frac{m\lambda^2}{2\hbar^2} < 0$$ one bound state energy

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Get $|B|$ by normalizing $\psi(x)$

$$1 \stackrel{!}{=} \int_{-\infty}^{\infty} dx\, |\psi(x)|^2 = 2|B|^2 \int_0^{\infty} dx\, e^{-2Kx} = \frac{|B|^2}{K}$$

$$\implies |B| = \sqrt{K} = \sqrt{\frac{m\lambda}{\hbar^2}} \implies B = |B| e^{i\alpha}$$

$$\psi(x) = \sqrt{\frac{m\lambda}{\hbar^2}} \exp\left(\frac{-m\lambda|x|}{\hbar^2}\right)$$ with $E = -\frac{m\lambda^2}{2\hbar^2}$

Now $E>0$ scattering states $\quad k \equiv \sqrt{\frac{2mE}{\hbar^2}}$ real $E>0$

$x<0$: $\psi_L'' = \frac{-2mE}{\hbar^2}\psi_L \Rightarrow \psi_L(x) = \underset{\text{right "moving"}}{\underset{\longrightarrow}{Ae^{ikx}}} + \underset{\text{left "moving"}}{\underset{\longleftarrow}{Be^{-ikx}}}$

$x>0$: $\psi_R'' = \frac{-2mE}{\hbar^2}\psi_R \Rightarrow \psi_R(x) = \underset{\longrightarrow}{Fe^{ikx}} + \cancel{\underset{\longleftarrow}{Ge^{-ikx}}}$

How many unknowns? $A, B, F, G, k(E)$ ← any k allowed

① ψ is continuous at $x=0$.

② ψ' has a jump discontinuity at $x=0$.

A "incoming", B - "reflected", F - "transmitted"

set $G=0$ no incoming waves from right.

divide by A, arbitrary

look at $\frac{B}{A}$ and $\frac{F}{A}$

① ψ is continuous: $\boxed{A+B=F}$

② ψ' has a jump discontinuity at $x=0$.

$$\psi_R'(0) = ikFe^{ikx}\Big|_{x=0} = ikF$$

$$\psi_L'(0) = ik\left(Ae^{ikx} - Be^{-ikx}\right)\Big|_{x=0} = ik(A-B)$$

$$\Delta\psi' = \frac{-2m\lambda}{\hbar^2}\psi(0) \qquad \text{same as before}$$

$$\boxed{ik(F-A+B) = \frac{-2m\lambda}{\hbar^2}F}$$

$$\Rightarrow \frac{B}{A} = \frac{i\frac{m\lambda}{\hbar^2 k}}{1-\frac{im\lambda}{\hbar^2 k}}, \qquad \frac{F}{A} = \frac{1}{1-\frac{im\lambda}{\hbar^2 k}}$$

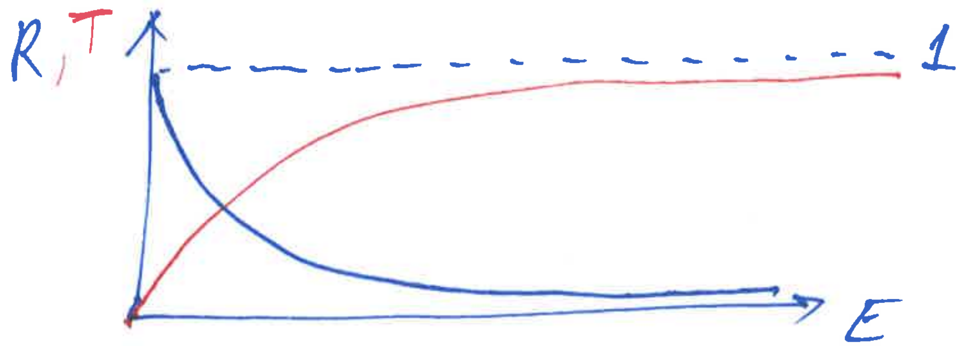
"Reflection" + "Transmission" Coefficient

Really for wave packets

$$R \equiv \frac{|B|^2}{|A|^2} = \frac{\left(\frac{m\lambda}{\hbar^2 k}\right)^2}{1+\left(\frac{m\lambda}{\hbar^2 k}\right)^2} \qquad T = \frac{|F|^2}{|A|^2} = \frac{1}{1+\left(\frac{m\lambda}{\hbar^2 k}\right)^2}$$

$$R+T=1$$

$$R = \frac{1}{1+\frac{2\hbar^2 E}{m\lambda^2}} \qquad T = \frac{1}{1+\frac{m\lambda^2}{2\hbar^2 E}}$$



Classically: $R=0$, $T=1$ for All $E>0$

The HW #4, Plot R+T for finite square well
R and T oscillate

Delta Function Barrier $V(x) = \lambda \delta(x)$, $\lambda > 0$

No bound states
non-denum. many scattering states

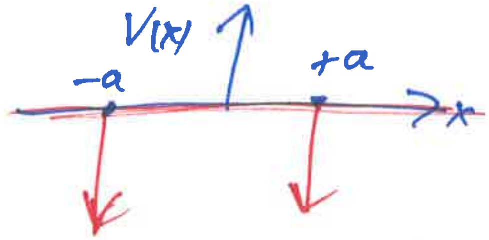
Classically: $R=1$, $T=0$

QM: T, R depend on $\lambda^2 \Rightarrow$ R and T
same as for the delta function well.

Tunneling phenomenon — Scanning tunneling microscopy, Sun shine.

Can show that $\langle \psi_{bound} | \psi_{scattering} \rangle = \emptyset$

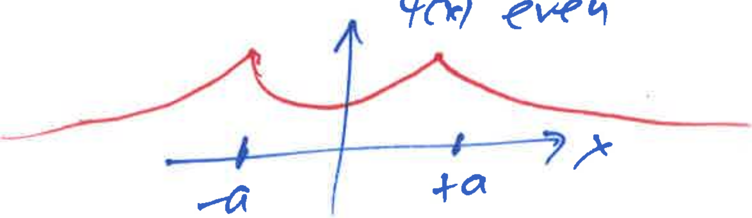
Double delta function Well



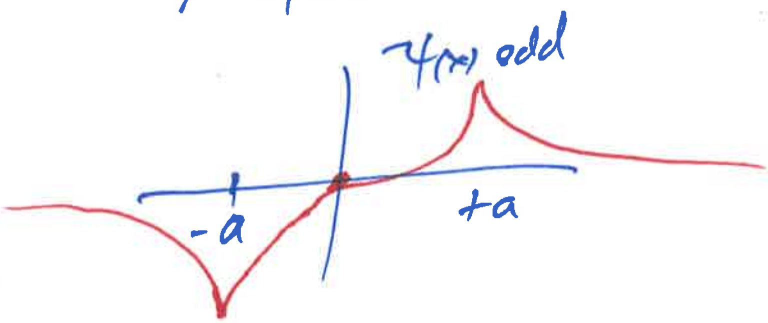
$V(x) = -\lambda\left[\delta(x+a) + \delta(x-a)\right]$

Non-denumerably many scattering states, $E > 0$

At least one bound state (even)



Maybe one more bound state (odd)

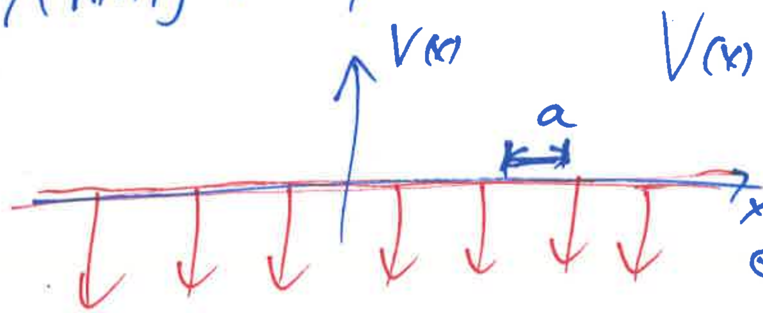


one: $\lambda \leq \frac{\hbar^2}{2ma}$

two: $\lambda \geq \frac{\hbar^2}{2ma}$

Kronig-Penney Model Solid-State Physics

$V(x) = -\lambda \sum_n \delta(x - na)$



$\leftarrow$ Dirac Comb

Felix Bloch's Theorem: $\psi(x+a) = e^{iqa}\psi(x)$

$\psi(x)$ not periodic / $|\psi(x+a)|^2 = |\psi(x)|^2$ is periodic

$\Rightarrow$ band structure