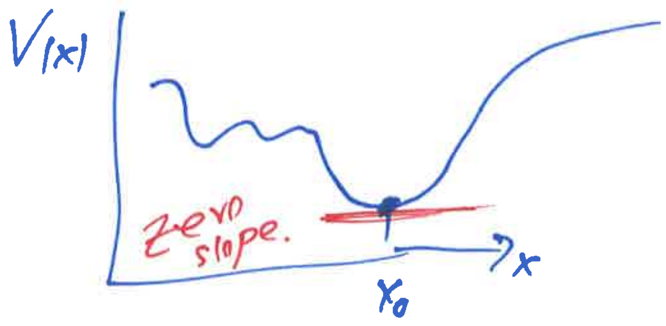
Quantum Harmonic Oscillator

Motivation: Any potential looks like H.O. near points of stable equilibrium



$$\left.\frac{dV}{dx}\right|_{x=x_0} = 0$$

Taylor Expansion:

$$V(x) = \underbrace{V(x_0)}_{\text{constant}} + \cancel{(x-x_0)\left.\frac{dV}{dx}\right|_{x=x_0}}^{\,0} + \frac{1}{2!}(x-x_0)^2 \underbrace{\left.\frac{d^2V}{dx^2}\right|_{x=x_0}}_{k>0} + \cancel{\ldots}^{\text{neglect}}$$

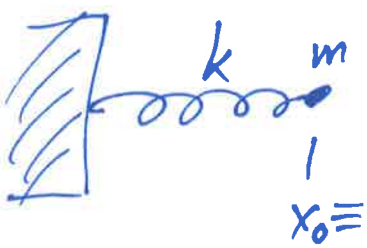
$k > 0$ minimum concave up

ideal Hooke's law spring

$$V(x) = \frac{1}{2}k(x-x_0)^2 + \cancel{\text{higher order}}^{\text{neglect}}$$

Classical Mechanics – Simple Harmonic Oscillator

$$\Sigma \vec{F} = m\vec{a}$$



no gravity
no friction

$$-kx(t) = m\frac{d^2x(t)}{dt^2}$$

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0$$

2nd-order linear, homo. DE
in x

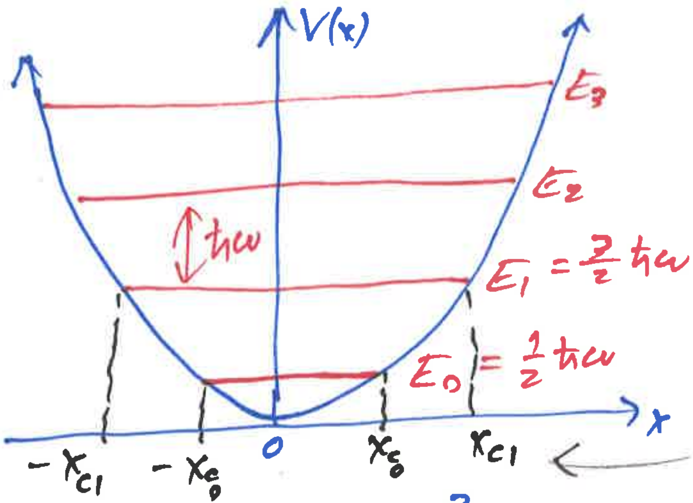
two linearly independent solutions

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}}, \quad k = m\omega^2$$

$$= Ce^{i\omega t} + De^{-i\omega t} = F\cos(\omega t + \varphi) = G\sin(\omega t + \beta)$$

get from initial conditions

QM $\quad V(x) = \frac{1}{2}m\omega^2 x^2$ $\qquad$ no scattering states



denumerably infinitely many bound states, $\aleph_0$

$\Rightarrow$ label states by integers.

$n = 0, 1, 2, 3, \ldots$

classical turning points

TISE: $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi(x) = E\psi(x)$$

2nd-order, linear in ψ, homogeneous

$\Rightarrow$ two linearly independent solutions $\quad$ one blows up as $x \to \pm\infty$.

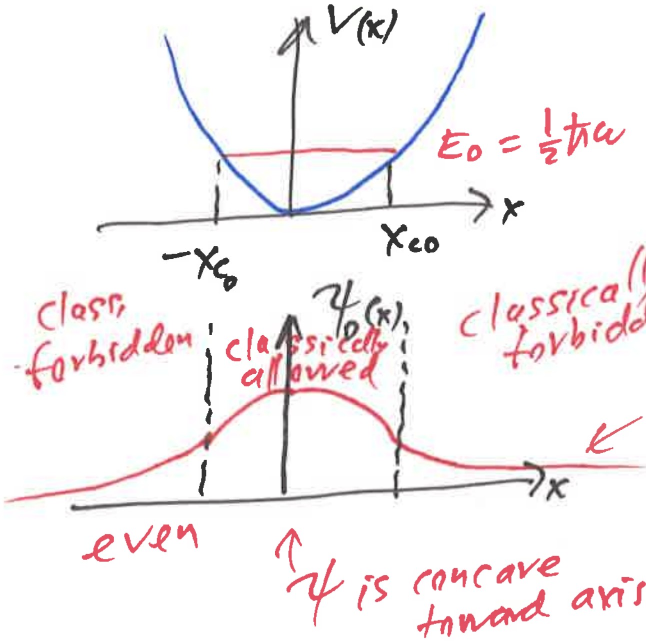
Energy eigenvalues: $E_n = \hbar\omega(n + \frac{1}{2})$

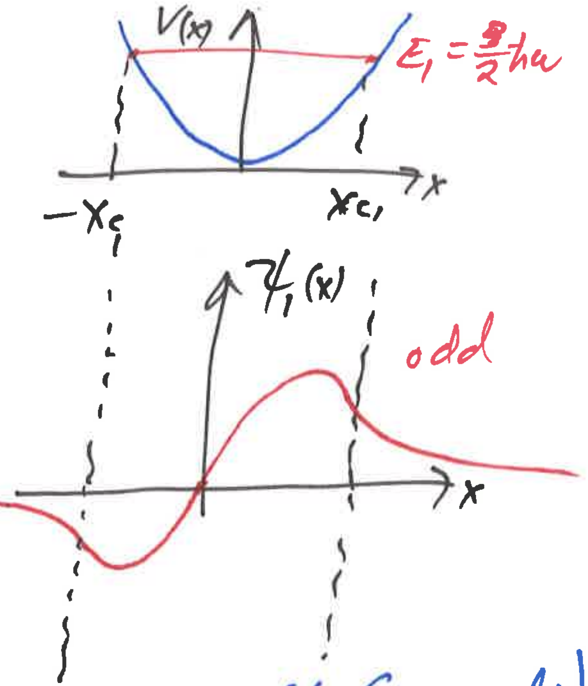
$$\psi_n(x) = \frac{1}{\sqrt{2^n \cdot n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right) H_n\underbrace{\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)}_{\equiv z \text{ dimensionless}}$$

Physics Hermite Polynomials (not probabilists')

$H_0(z) = 1, \quad H_1(z) = 2z, \quad H_2(z) = 4z^2 - 2, \quad H_3(z) = 8z^3 - 12z$

- coefficient highest power of z is 2^n
- n even (odd) $\Rightarrow$ only even (odd) powers of z





If $V(x)$ is even, then ψ has definite parity (even, odd)

Classical turning points

x: $E_n < V(x)$

$$E_0 < V(x) = \frac{1}{2} m\omega^2 x^2$$

$$\cancel{\tfrac{1}{2}}\hbar\omega < \cancel{\tfrac{1}{2}} m\omega^2 x_{0c}^2$$

$$x_{c_0} = \sqrt{\frac{\hbar}{m\omega}}$$

$$E_1 < V(x)$$

$$\frac{3}{2}\hbar\omega < \frac{1}{2} m\omega^2 x_{c_1}^2$$

$$x_{c_1} = \sqrt{\frac{3\hbar}{m\omega}} > x_{c_0}$$

---

orthogonality

$$\int_{-\infty}^{+\infty} dz\, e^{-z^2} H_n(z) H_p(z) = \delta_{np} \sqrt{\pi}\, n!\, 2^n$$

$\uparrow$ weight

Ladder Operators aka Raising + Lowering Ops aka Creation + Annihilation Ops

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = E\,\psi(x)$$ coordinate basis

$$\frac{\hat{p}^2}{2m}|\psi\rangle + \frac{1}{2}m\omega^2\hat{x}^2|\psi\rangle = E|\psi\rangle$$ ket space

Hamiltonian $\hat{H} = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]$, $\hat{H}|\psi\rangle = E|\psi\rangle$

Define : $\hat{a} \equiv \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p})$ lowering operator

$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} - i\hat{p})$ raising operator

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a})$, $\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^\dagger - \hat{a})$

Given the fundamental commutation relation $[\hat{x},\hat{p}] = i\hbar\hat{\mathbb{1}}$
$[\hat{x},\hat{x}] = 0 = [\hat{p},\hat{p}]$

what is $[\hat{a},\hat{a}^\dagger]$?

$$= \frac{1}{2\hbar m\omega}\left[(m\omega\hat{x} + i\hat{p}), (m\omega\hat{x} - i\hat{p})\right]$$

$$= \frac{1}{2\hbar m\omega}\left(im\omega\underbrace{[\hat{p},\hat{x}]}_{-i\hbar} - im\omega\underbrace{[\hat{x},\hat{p}]}_{i\hbar}\right) = \hat{\mathbb{1}}$$

$$\boxed{[\hat{a}^\dagger,\hat{a}] = -\hat{\mathbb{1}}}$$

$$\hat{a}^\dagger \hat{a} = \frac{1}{(2\hbar m\omega)}(-i\hat{p} + m\omega\hat{x})(i\hat{p} + m\omega\hat{x})$$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p}^2 + m^2\omega^2\hat{x}^2 - im\omega\,\hat{p}\hat{x} + im\omega\,\hat{x}\hat{p}\right)$$

$$= \frac{1}{2\hbar m\omega}\left(\hat{p}^2 + m^2\omega^2\hat{x}^2 + im\omega\underbrace{[\hat{x},\hat{p}]}_{i\hbar\hat{\mathbb{I}}}\right)$$

$$= \frac{1}{\hbar\omega}\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\right) - \frac{1}{2}\hat{\mathbb{I}} = \frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\hat{\mathbb{I}}$$

$$\Rightarrow \quad \hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\hat{a}\hat{a}^\dagger = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\hat{\mathbb{I}} \quad \Rightarrow \quad \hat{H} = \hbar\omega\left(\hat{a}\hat{a}^\dagger - \frac{1}{2}\right)$$

Shorthand $|\psi_n\rangle = |n\rangle$ where $\langle x|\psi_n\rangle = \psi_n(x)$

$$\hat{H}\,|n\rangle = E_n|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

$$\parallel$$

$$\hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle = \hbar\omega\,\hat{a}^\dagger\hat{a}|n\rangle + \frac{1}{2}\hbar\omega|n\rangle$$

$$\Rightarrow \quad \hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle$$

Define $\hat{N} = \hat{a}^\dagger\hat{a}$ number operator

$\hat{N}|n\rangle = n|n\rangle$, $[\hat{N},\hat{a}] = -\hat{a}$, $[\hat{N},\hat{a}^\dagger] = +\hat{a}^\dagger$

Why raising + Lowering ?

$\langle\varphi| = \langle n|\hat{a}^{\dagger\dagger} = \langle n|\hat{a}$

Consider a new ket $|\varphi\rangle = \hat{a}^{\dagger}|n\rangle$

What is $|\varphi\rangle$? Energy eigenstate? Eigenvalue?

$$\hat{H}|\varphi\rangle = \left[\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\tfrac{1}{2}\right)\right]\left(\hat{a}^{\dagger}|n\rangle\right)$$

$$= \hbar\omega\left(\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} + \tfrac{1}{2}\hat{a}^{\dagger}\right)|n\rangle$$

use $[\hat{a},\hat{a}^{\dagger}] = \hat{\mathbb{1}}$

$\hat{a}\hat{a}^{\dagger} = \hat{\mathbb{1}} + \hat{a}^{\dagger}\hat{a}$

$$= \hbar\omega\,\hat{a}^{\dagger}\left(\hat{a}\hat{a}^{\dagger} + \tfrac{1}{2}\right)|n\rangle$$

$$= \hbar\omega\,\hat{a}^{\dagger}\Big(\underbrace{\hat{a}^{\dagger}\hat{a} + 1 + \tfrac{1}{2}}_{\hat{H}}\Big)|n\rangle$$

$\hat{H}|n\rangle = E_n|n\rangle$

$$= \hat{a}^{\dagger}\left(\hat{H} + \hbar\omega\right)|n\rangle = \hat{a}^{\dagger}\left(E_n + \hbar\omega\right)|n\rangle$$

$$= \left(E_n + \hbar\omega\right)\hat{a}^{\dagger}|n\rangle = \left(E_n + \hbar\omega\right)|\varphi\rangle$$

$$= E_{n+1}|\varphi\rangle$$

$|\varphi\rangle$ is an energy eigenket one rung higher.

$\langle n|n\rangle = 1,\quad \langle p|n\rangle = 0$

$|\varphi\rangle$ is not normalized

$$\langle\varphi|\varphi\rangle = \langle n|\hat{a}\hat{a}^{\dagger}|n\rangle = \langle n|\left(\hat{a}^{\dagger}\hat{a} + \hat{\mathbb{1}}\right)|n\rangle = \langle n|\left(\hat{N} + \hat{\mathbb{1}}\right)|n\rangle$$

$$= n\langle n|n\rangle + \langle n|n\rangle = (n+1)$$

$$|\varphi\rangle = \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|(n+1)\rangle,\quad |n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^{\dagger}|n\rangle$$