Similarly $|x\rangle \equiv \hat{a}|n\rangle$ is also an energy eigenstate

$$\hat{H}|x\rangle = \hat{H}\hat{a}|n\rangle = (E_n - \hbar\omega)|x\rangle \Rightarrow |x\rangle \propto |(n-1)\rangle$$

again $|x\rangle$ is not normalized $\langle x|x\rangle \neq 1$

$$\langle x|x\rangle = \left(\langle n|\hat{a}^{+}\right)\left(\hat{a}|n\rangle\right) = \langle n|\hat{N}|n\rangle = \langle n|n\hat{\mathbb{I}}|n\rangle$$
$$= n\langle n|n\rangle = n$$

$$|x\rangle = \hat{a}|n\rangle = \sqrt{n}\,|(n-1)\rangle$$

There is no highest rung on the ladder: $(\hat{a}^{+})^{p}|n\rangle, \ldots$

There is, however, a lowest rung (see $\mathcal{M}_{\text{III}}$)

$$\langle n|\hat{N}|n\rangle = n\langle n|n\rangle = \underline{n} = \left(\langle n|\hat{a}^{+}\right)\left(\hat{a}|n\rangle\right)$$
$$= \|\hat{a}|n\rangle\|^2 = \||x\rangle\|^2 \underline{\geq 0} \Rightarrow n \geq 0$$

Call $|0\rangle$ the lowest state on the ladder

the ground state..

$|0\rangle$ is <u>not</u> the zero ket, nor is it the QFT vacuum.

What is $\hat{a}|0\rangle$? Must be $\hat{a}|0\rangle = \emptyset$ ← number zero

Ground state wave function

$\langle x | 0 \rangle \equiv \psi_0(x)$

Coordinate basis:

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega x + \hbar \frac{d}{dx}\right)\psi_0(x) = 0$$

$$\Rightarrow \frac{d\psi_0(x)}{dx} = -\frac{m\omega}{\hbar} x\, \psi_0(x)$$

1st-order linear in ψ_0 homogeneous D.E.

$$\Rightarrow \frac{d\psi}{\psi} = -\frac{m\omega}{\hbar} x\, dx$$

integrate

$$\ln(\psi_0) = \frac{-m\omega}{2\hbar} x^2 + C$$

exponentiate

$$\psi_0(x) = A \exp\left(-\frac{m\omega}{2\hbar} x^2\right)$$

get A by normalizing $\int_{-\infty}^{+\infty} dx\, |\psi_0(x)|^2 = 1 \Rightarrow A = \sqrt[4]{\frac{m\omega}{\pi\hbar}}$

$$\boxed{\psi_0(x) = \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(\frac{-m\omega}{2\hbar} x^2\right) = \langle x | 0 \rangle \quad \text{gaussian}}$$

Energy $\hat{H}|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \underline{\frac{1}{2}\hbar\omega}\, |0\rangle = E_0 |0\rangle$

Now can generate all the higher states with $\hat{a}^\dagger$:

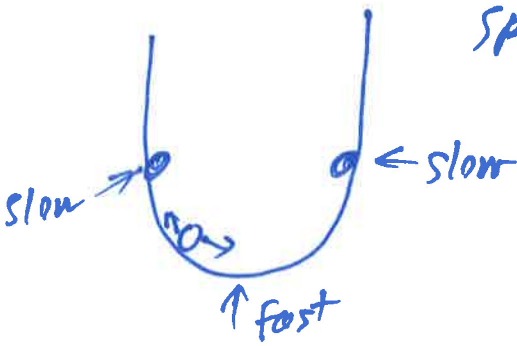
$$|n\rangle = \frac{1}{\sqrt{n!}}\left(\hat{a}^\dagger\right)^n |0\rangle \quad , \quad \langle n | n \rangle = 1$$

3-dimensional QHO

$$\langle \vec{r} \,|\, n, p, q \rangle = \psi_n(x)\,\psi_p(y)\,\psi_q(z)$$

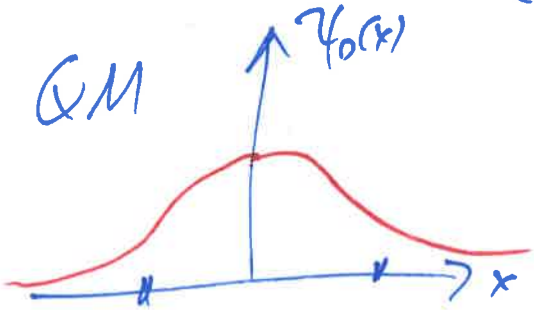
with energy $E_{npq} = \hbar\omega\left(n+p+q+\frac{3}{2}\right)$

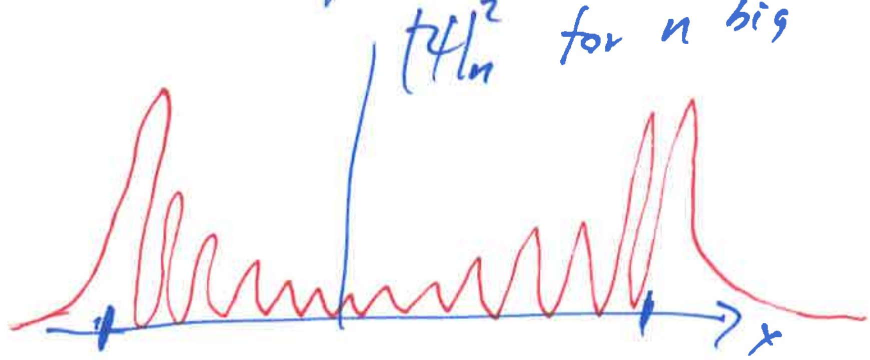
Classically, where does a marble rolling in parabolic well (or mass on a spring)  spend most of its time?



classical marble ~ very large n quantum number.

QM





Coherent States

Only exist for the QHO.

ground state $|0\rangle$ is the only energy eigenstate that saturates the uncertainty principle.

$$\sigma_{x_0}\sigma_{p_0} = \Delta x_0 \cdot \Delta p_0 = \frac{\hbar}{2}$$

For higher states: $\sigma_{x_n} \cdot \sigma_{p_n} = \underbrace{\boxed{}}^{HW} > \frac{\hbar}{2}$

But linear combinations of energy eigenstates called "coherent states" saturate the uncertainty inequality relation $\forall$ time t. They happen to be eigenstates of the lowering operator $\hat{a}$.

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

where α is any complex number.

There are no normalizable eigenkets of the raising operator $\hat{a}^+$

$$\left(m\omega x - \hbar\frac{d}{dx}\right)\varphi(x) = \alpha\,\varphi(x) \Rightarrow \varphi(x) = A\exp\left(\frac{m\omega x^2}{\hbar} - \frac{\alpha x}{\hbar}\right)$$

Not normalizable $\Rightarrow$ not a bound state.

Expand the coherent states in energy eigenstates

$$|\alpha_0\rangle = \hat{\mathbb{1}}\,|\alpha\rangle = \left(\sum_{n=0}^{\infty} |n\rangle\langle n|\right)|\alpha\rangle$$

$$= \sum_{n=0}^{\infty} |n\rangle \underbrace{\langle n|\alpha\rangle}_{c_n} = \sum_{n=0}^{\infty} c_n\,|n\rangle$$

HW

Add in the time dependence.

$$E_n = (n+\tfrac{1}{2})\hbar\omega$$

$$|\alpha(t)\rangle = \sum_{n=0}^{\infty} c_n \cdot e^{-\frac{iE_n t}{\hbar}}\,|n\rangle$$

Generic state $|\psi\rangle = \sum_{n=0}^{\infty} b_n\,|n\rangle$