$\underline{\ell = \frac{1}{2}}$ basis: $\left\{ |\tfrac{1}{2},+\tfrac{1}{2}\rangle_z, |\tfrac{1}{2},-\tfrac{1}{2}\rangle_z \right\} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}_z, \begin{pmatrix}0\\1\end{pmatrix}_z \right\}$

$= \{\text{spin up, spin down}\} = \{|\uparrow\rangle, |\downarrow\rangle\} = \{|+\rangle, |-\rangle\}$

No coordinate space representation.

$\hat{S}_3 = \hat{S}_z$

$\hat{L}_3 = \hat{L}_z = \frac{\hbar}{2}\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2}\sigma_3 = \frac{\hbar}{2}\sigma_z$ Pauli matrix

$\hat{S}_+ = \hat{L}_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \hat{L}_+\begin{pmatrix}1\\0\end{pmatrix} = 0$

$\hat{L}_+\begin{pmatrix}0\\1\end{pmatrix} = \hbar\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \hat{L}_+|-\rangle = \hbar|+\rangle$

$\hat{S}_- = \hat{L}_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\hat{L}_+|\ell m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\;|\ell, m+1\rangle$

$\hat{L}_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{3}{2}) - (-\tfrac{1}{2})(\tfrac{1}{2})}\;|\tfrac{1}{2},\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2},\tfrac{1}{2}\rangle$

$\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_1$

$\hat{L}_y = \frac{1}{2i}(\hat{L}_+ - \hat{L}_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_2$

$\hat{L}_x, \hat{L}_y, \hat{L}_z, \hat{L}^2$ are hermitian $\hat{L}_i^\dagger = \hat{L}_i$

$\hat{S}^2 = \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{3}{4}\hbar^2 \mathbb{I}$

$\ell(\ell+1) = \frac{1}{2}(\frac{3}{2}) = \frac{3}{4}$

$\underline{l=1}$ basis vectors

$$\{|1,1\rangle, \underbrace{|1,0\rangle}_{|lm\rangle}, |1,-1\rangle\} = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$$

$$\hat{L}_z = \hat{L}_3 = \hbar\begin{pmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{pmatrix}$$

$$\hat{L}_+ = \hbar\begin{pmatrix}0 & \sqrt{2} & 0\\0 & 0 & \sqrt{2}\\0 & 0 & 0\end{pmatrix}, \quad \hat{L}_- = \hat{L}_+^\dagger = \hbar\begin{pmatrix}0 & 0 & 0\\\sqrt{2} & 0 & 0\\0 & \sqrt{2} & 0\end{pmatrix}$$

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{pmatrix}$$

$$\hat{L}^2 = 2\hbar^2\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix} = 2\hbar^2 \mathbb{1}_{3\times 3} = l(l+1)\Big|_{l=1} \hbar^2 \mathbb{1}$$

In coordinate basis, eigenfunctions are $Y_{lm}(\theta,\varphi)$

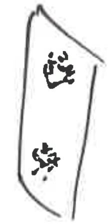
$$Y_{11}(\theta,\varphi), \quad Y_{10}(\theta,\varphi), \quad Y_{1-1}(\theta,\varphi)$$

$$\begin{array}{c} Y_{11}(\theta,\varphi) \\ \parallel \\ \langle \vec{r}\,|1,1\rangle \end{array}$$

Stern-Gerlach expt

silver atoms $\longrightarrow$ 

inhomogeneous B field



electrons would experience Lorentz force $F = q(\vec{E} + \vec{v}\times\vec{B})$

Ag has single outer e^-, unpaired spin, no orbital ang. momentum

if $\vec{B}$ field were homogeneous, $\exists$ torque on magnetic dipoles but no force.

$$\vec{F} = -\vec{\nabla}(U) = +\vec{\nabla}(\vec{\mu}\cdot\vec{B})$$

Uhlenbeck + Goudsmit – new quantum like angular momentum
mid 20's

Advisor – Ehrenfest

H. Lorentz reviewer

$\bar{e}$ $\bigcirc R_c$ also solves "anomalous" Zeeman effect

even # of states in a B field

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orthogonality + completeness for $l = \frac{1}{2}$

$$\langle + | + \rangle = 1 = \langle - | - \rangle \quad , \quad \langle + | - \rangle = 0$$

$$\mathbb{1} = |+\rangle\langle +| + |-\rangle\langle -|$$

For <u>orbital</u> angular momentum, ℓ must be integral

$$\hat{L}_z\,|\ell,m\rangle = m\hbar\,|\ell,m\rangle \Longrightarrow \frac{\hbar}{i}\frac{\partial}{\partial\varphi}Y_{\ell m}(\theta,\varphi) = m\hbar\, Y_{\ell m}(\theta,\varphi)$$

$Y_{\ell m}(\theta,\varphi)$ must be continuous, not because that is a postulate of QM, but because if $Y_{\ell m}(\theta,\varphi)$ were discontinuous at (say) $\varphi=0$, then

$$\left.\frac{\partial}{\partial\varphi}Y_{\ell m}(\theta,\varphi)\right|_{\varphi=0} = \delta(\varphi)\Big|_{\varphi=0}$$

what's wrong with that? Incompatible with $Y_{\ell m}$'s do not contain Dirac delta functions.

$$Y_{\ell m}(\theta,\underset{\text{say}}{\varphi=0}) = Y_{\ell m}(\theta,\varphi=2\pi) \Rightarrow \text{azimuthal equation } L12$$

$$e^{im\varphi} = e^{0} = 1 = e^{2\pi i m} \Rightarrow$$ m must be integral $\Downarrow$ ℓ must be integral.

$$\boxed{e^{2\pi i m} = -1 \text{ if } m \text{ is half integral}}$$

Back to hydrogen

In any central potential $V(\vec{r}) = V(r) = V(|\vec{r}|)$,
$\vec{L}$ is conserved: $[\hat{H}, \hat{\vec{L}}] = 0$

But for hydrogen only $V(r) = \dfrac{-e^2}{4\pi\epsilon_0 r} \propto \dfrac{1}{r}$
there is another conserved vector

Laplace-Runge-Lenz vector:

$$\hat{\vec{A}} \equiv \frac{\hat{\vec{p}} \times \hat{\vec{L}} - \hat{\vec{L}} \times \hat{\vec{p}}}{2m} + V(\hat{R})\,\hat{\vec{R}}$$

$$[\hat{H}, \hat{L}_i] = 0 \Rightarrow \vec{L} \text{ is conserved, } \hat{H} \text{ is a scalar operator}$$

$$[\hat{H}, \hat{A}_i] = 0 \Rightarrow \vec{A} \text{ is conserved}$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{L}_k \Rightarrow \vec{L} \text{ is vector operator}$$

Lie algebra $= so(3) \simeq su(2)$

$$[\hat{L}_i, \hat{A}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{A}_k \Rightarrow \vec{A} \text{ is vector op.}$$

$$[\hat{A}_i, \hat{A}_j] = \frac{\hbar}{i} \sum_{k=1}^{3} \epsilon_{ijk} \hat{L}_k \frac{2}{m} \hat{H} \Rightarrow ?$$

$$\hat{A}^2 = \hat{\vec{A}} \cdot \hat{\vec{A}} = \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \mathbb{1} + \frac{2}{m} \hat{H} (\hat{L}^2 + \hbar^2)$$

$$\hat{\vec{A}} \cdot \hat{\vec{L}} = 0 = \hat{\vec{L}} \cdot \hat{\vec{A}}$$

$E<0$ is an energy eigenvalue

Need to be in that eigen subspace.

Define $\hat{T}_i \equiv \frac{1}{2}\left(\hat{L}_i + \sqrt{\frac{m}{-2E}}\,\hat{A}_i\right)$

$$\hat{S}_i \equiv \frac{1}{2}\left(\hat{L}_i - \sqrt{\frac{m}{-2E}}\,\hat{A}_i\right)$$

$$[\hat{T}_i, \hat{S}_j] = 0$$

$$[\hat{T}_i, \hat{T}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{T}_k \Rightarrow su(2)$$

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{S}_k \Rightarrow su(2)$$

$$su(2) \oplus su(2) = so(4) \quad \text{Lie algebras}$$

$\uparrow$ direct sum $\leftarrow$ means independent in physics