Rotations

Coordinate rotations $\vec{r}\,' = \underline{\underline{R}}\ \vec{r}$

$\underline{\underline{R}}$: 3x3 matrix

$\vec{r}$: 3x1 vector

Lie Group $SO(3)$

Lie Algebra $= so(3) = su(2)$

$\underline{\underline{R}}$ is orthogonal: $\underline{\underline{R}}^T = \underline{\underline{R}}^{-1}$

$$\langle \vec{r}\,' | \hat{R} | \psi \rangle = \psi'(\vec{r}\,') = \psi(\vec{r}) = \psi(\underline{\underline{R}}^{-1}\vec{r}\,')$$

Book: $\psi'(\vec{r}) = \psi(\underline{\underline{R}}^{-1}\vec{r})$

Rotation by an infinitesimal angle $d\alpha$ about the direction $\hat{u}$ ← unit vector $\hat{u}\cdot\hat{u} = 1$

$$\underline{\underline{R}}_u(d\alpha)\,\vec{v} = \left(\underline{\underline{1}} + d\alpha\, \hat{u}\times\right)\vec{v}$$

Let $\hat{u} = \hat{z}$

$$\underline{\underline{R}}_z^{-1}(d\alpha)\,\vec{r} = \vec{r} - d\alpha\, \hat{z}\times\vec{r} = \begin{cases} (x + y\,d\alpha)\,\hat{x} + \\ (y - x\,d\alpha)\,\hat{y} + \\ z\,\hat{z} \end{cases}$$

$$\psi'(x,y,z) = \psi(x + y\,d\alpha,\ y - x\,d\alpha,\ z)$$

$$= \psi(x,y,z) + d\alpha\left[y\frac{\partial\psi}{\partial x} - x\frac{\partial\psi}{\partial y}\right] + \cancel{O(d\alpha^2)}$$

neglect

$$= \psi(x,y,z) - d\alpha\left[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right]\psi(x,y,z)$$

$$\psi'(x,y,z) = \underbrace{\left[\hat{\mathbb{1}} - \frac{i}{\hbar} d\alpha \, \hat{L}_z\right]}_{\hat{R}_z(d\alpha)} \psi(x,y,z) \equiv \hat{R}_z(d\alpha)\psi(x,y,z)$$

$$\hat{L}_z = \hat{\vec{L}} \cdot \hat{z}$$

same for x, y components

$$\hat{R}_u(d\alpha) = \hat{\mathbb{1}} - \frac{i}{\hbar} d\alpha \, \hat{\vec{L}} \cdot \hat{u} \qquad \text{for infinitesimal } d\alpha$$

$$\underbrace{\hat{R}_z(\alpha + d\alpha) = \hat{R}_z(\alpha)\hat{R}_z(d\alpha)}_{\text{Group property}} = \hat{R}_z(\alpha)\left(\hat{\mathbb{1}} - \frac{i}{\hbar} d\alpha \, \hat{L}_z\right)$$

$$d\hat{R}_z = \hat{R}_z(\alpha + d\alpha) - \hat{R}_z(\alpha) = -\frac{i}{\hbar} d\alpha \, \hat{R}_z(\alpha)\hat{L}_z \quad \text{integrate}$$

$$\boxed{\hat{R}_z(\alpha) = \exp\left[-\frac{i}{\hbar}\alpha \hat{L}_z\right]}$$

$$\frac{\partial}{\partial\varphi}$$

$$\hat{R}_u(\alpha) = \exp\left[-\frac{i}{\hbar}\alpha \, \hat{\vec{L}} \cdot \hat{u}\right]$$

Lie algebra generator $\hat{L}$ when exponentiated becomes the Lie group element $\hat{R}$

We saw this before – HW #3

$$\exp\left[-\frac{i}{\hbar}\hat{p}_x a\right] \quad \text{generates spatial translations}$$

$\frac{\partial}{\partial x}$

$$\hat{X}\left(e^{-\frac{i}{\hbar}\hat{p}_x a}|x\rangle\right) = (x+a)\left(e^{-\frac{i}{\hbar}\hat{p}_x a}|x\rangle\right)$$

$$e^{-\frac{i}{\hbar}\hat{p}_x a}|x\rangle = |x+a\rangle$$

Also $\hat{U}(t,0) = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]$

generates time translations

$$\hat{U}(t,0)\,|\psi(0)\rangle = |\psi(t)\rangle$$

All the operators in the exponentials are hermitian: $\hat{L} = \hat{L}^\dagger$, $\hat{H} = \hat{H}^\dagger$, $\hat{p} = \hat{p}^\dagger$

All the generators are unitary $\hat{U}^\dagger = \hat{U}^{-1}$

$$\hat{R}_u^\dagger(\alpha) = \hat{R}_u(\alpha)^{-1}$$

Rotations of Observables

old news $\langle \psi(0) | \underbrace{\hat{U}^{\dagger}_{(t,0)} \hat{A}_s \hat{U}(t,0)}_{\hat{A}_H} | \psi(0) \rangle$

$|\psi_n'\rangle = \hat{R} |\psi_n\rangle$

$\hat{B} |\psi_n\rangle = b_n |\psi_n\rangle$

$\hat{B}' |\psi_n'\rangle = b_n |\psi_n'\rangle$

$\hat{R}^{-1} = \hat{R}^{\dagger}$ unitary: $\hat{B}' \hat{R} |\psi_n\rangle = b_n \hat{R} |\psi_n\rangle$

$\hat{R}^{\dagger} \hat{B}' \hat{R} |\psi_n\rangle = b_n \hat{\mathbb{1}} |\psi_n\rangle$

$$\hat{B} = \hat{R}^{\dagger} \hat{B}' \hat{R} \implies \boxed{\hat{B}' = \hat{R} \hat{B} \hat{R}^{\dagger}}$$

Infinitesimal rotations

$$\hat{B}' = \left[\hat{\mathbb{1}} - \frac{i}{\hbar} d\alpha\, \hat{\vec{L}} \cdot \hat{\vec{u}}\right] \hat{B} \left[\hat{\mathbb{1}} + \frac{i}{\hbar} d\alpha\, \hat{\vec{L}} \cdot \hat{\vec{u}}\right]$$

$$= \hat{B} - \frac{i}{\hbar} d\alpha \left[\hat{\vec{L}} \cdot \hat{\vec{u}}, \hat{B}\right]$$

Scalar operator e.g. $\hat{H}$

$$\hat{B}' = \hat{B} \Longrightarrow [\hat{\vec{L}}, \hat{B}] = 0 \quad \left\{ \begin{array}{l} [\hat{B}, \hat{L}_x] = 0 \\ [\hat{B}, \hat{L}_y] = 0 \\ [\hat{B}, \hat{L}_z] = 0 \end{array} \right.$$

$$[\hat{A}, \hat{\vec{L}}] = 0$$

Vector operator e.g. $\hat{\vec{A}}$ $\quad [\hat{L}_i, \hat{A}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{A}_k$

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spin : $\frac{1}{2}\hbar$

$|+\rangle_z$ and $|-\rangle_z$ form a basis of the 2-dimensional Hilbert space.

General ket $|\psi\rangle = \alpha |+\rangle_z + \beta |-\rangle_z$ with $|\alpha|^2 + |\beta|^2 = 1$

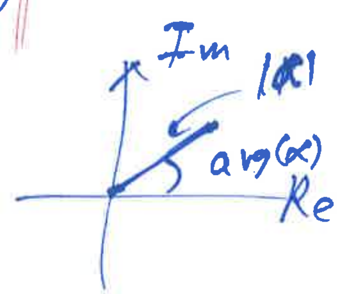
$$|\alpha| \geq 0 , \ |\beta| \geq 0 \Longleftarrow 0 \leq \theta \leq \pi$$

define $|\alpha| = \cos\left(\frac{\theta}{2}\right)$, $|\beta| = \sin\left(\frac{\theta}{2}\right)$

define also $\varphi = \arg(\alpha) - \arg(\beta)$ $\quad$ $\arg(\alpha) = \frac{1}{2}(\chi - \varphi)$

$\chi = \arg(\alpha) + \arg(\beta)$ $\quad$ $\arg(\beta) = \frac{1}{2}(\chi + \varphi)$

$$\alpha = |\alpha| e^{i \arg(\alpha)}$$



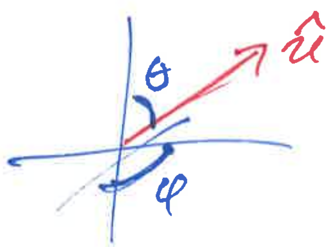
$$|\psi\rangle = \alpha|+\rangle_z + \beta|-\rangle_z = |\alpha| e^{i\arg(\alpha)} |+\rangle_z + |\beta| e^{i\arg(\beta)} |-\rangle_z$$

$$= \cos\left(\frac{\theta}{2}\right) e^{i\frac{\chi}{2}} e^{-i\frac{\varphi}{2}} |+\rangle_z + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\chi}{2}} \cdot e^{i\frac{\varphi}{2}} |-\rangle_z$$

$$= \underbrace{e^{i\frac{\chi}{2}}}_{\text{overall phase, not measurable.}} \underbrace{\left[\cos\left(\frac{\theta}{2}\right) e^{-i\frac{\varphi}{2}} |+\rangle_z + \sin\left(\frac{\theta}{2}\right) e^{i\frac{\varphi}{2}} |-\rangle_z\right]}_{|+\rangle_u}$$

To prepare this state:

Measure spin along the $\hat{u}$ direction until you obtain $+\frac{\hbar}{2}$



$${}_u\langle +|+\rangle_u = 1 \qquad {}_u\langle -|-\rangle_u$$

$${}_u\langle +|-\rangle_u = 0$$

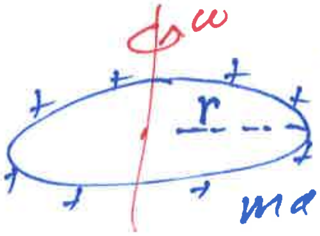
$$|\psi\rangle = e^{i\frac{\chi}{2}} |+\rangle_u$$

Larmor precession

Magnetic dipole moment of the electron

$T = \frac{2\pi}{\omega}$ = period

Model



q = charge

radius = r

mass = m

current

magnetic dipole moment: $\mu = I \cdot \text{Area}$

$= \frac{q}{T} \cdot \pi r^2$

angular momentum $\vec{L}$ = Spin $= I\omega = mr^2 \frac{2\pi}{T}$

$= S =$ I moment of inertia

gyromagnetic ratio $\gamma = \frac{\mu}{S} = \frac{q}{2m}$

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spherical shell or solid ball, γ is the same as the ring

as long as the charge and mass are distributed uniformly.

$\vec{\mu} = \frac{q}{2m}\vec{S}$, for an electron: $\vec{\mu}_e = \frac{-e}{2m_e}\vec{S}$

but in fact $\boxed{\vec{\mu}_e = \frac{-e}{m_e}\vec{S}}$

The factor of 2 difference is from the Dirac equation, which incorporates spin $\frac{1}{2}$ correctly.