3 New Postulates for Spin in QM

- The spin operator $\hat{\vec{S}}$ is an angular momentum

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_{k=1}^{3} \epsilon_{ijk} \hat{S}_k$$

su(2) Lie algebra

- The spin operators acts in a new finite-dimensional Hilbert space (two-dimensional for $s = \frac{1}{2}$) ($2s+1$ in general)

$$\hat{S}^2 = \hat{\vec{S}} \cdot \hat{\vec{S}} \Rightarrow \hat{S}^2 |s, m_s\rangle = s(s+1)\hbar^2 |s, m_s\rangle$$

$$\hat{S}_z |s, m_s\rangle = m_s \hbar |s, m_s\rangle$$

diagonal (in this rep) but not $\propto \hat{\mathbb{I}}$

$$[\hat{S}_z, \hat{S}_x] \neq 0 \neq [\hat{S}_z, \hat{S}_y]$$

- the state space of a particle with spin is the <u>tensor product</u>

$$\mathcal{E} = \mathcal{E}_r \otimes \mathcal{E}_s \Rightarrow$$ all spin observables in $\mathcal{E}_s$ commute with all the observables in $\mathcal{E}_r$.

Complete set of commuting observables CSCO

$\{\hat{X}, \hat{Y}, \hat{Z}, \hat{S}^2, \hat{S}_y\}$, $\{\hat{P}_x, \hat{P}_y, \hat{P}_z, \hat{S}^2, \hat{S}_x\}$

$\{\hat{H}, \hat{L}^2, \hat{L}_z, \hat{S}^2, \hat{S}_z\}$ label state with eigenval. $|n, \ell, m_\ell, s, m_s\rangle$

Total Angular Momentum Operator

$$\hat{\vec{J}} \equiv \hat{\vec{L}} + \hat{\vec{S}} \quad , \quad [\hat{J}_i , \hat{J}_k] = i\hbar \sum_{\ell=1}^{3} \epsilon_{ik\ell} \hat{J}_\ell$$

$$\hat{R}_u(\alpha) = \exp\left[\frac{-i}{\hbar}\alpha \hat{\vec{J}} \cdot \hat{\vec{u}}\right] \equiv {}^{(r)}\hat{R}_u(\alpha) \otimes {}^{(S)}\hat{R}_u(\alpha)$$

$$\exp\left[\frac{-i}{\hbar}\alpha \hat{\vec{L}} \cdot \hat{\vec{u}}\right] \otimes \exp\left[\frac{-i}{\hbar}\alpha \hat{\vec{S}} \cdot \hat{\vec{u}}\right]$$

in $\mathcal{E}$

If $|\psi\rangle = {}^{(r)}|\varphi\rangle \otimes {}^{(S)}|\chi\rangle$ then

$$|\psi'\rangle = \hat{R}_u(\alpha)|\psi\rangle = \left[{}^{(r)}\hat{R}_u(\alpha)|\varphi\rangle\right] \otimes \left[{}^{(S)}\hat{R}_u(\alpha)|\chi\rangle\right]$$

If $s = \frac{1}{2}$, the $|\chi\rangle$ has two components $\begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix}$ spinor

$$\boxed{2\times 2} \quad {}^{(S)}\hat{R}_u(\alpha) = \exp\left[\frac{-i}{\hbar}\alpha \hat{\vec{S}} \cdot \hat{\vec{u}}\right] = \exp\left[-i\frac{\alpha}{2}\vec{\sigma} \cdot \hat{\vec{u}}\right]$$

$$= \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\,\vec{\sigma} \cdot \hat{\vec{u}} \sin\left(\frac{\alpha}{2}\right)$$

$$\begin{bmatrix} \cos\left(\frac{\alpha}{2}\right) - i u_z \sin\left(\frac{\alpha}{2}\right) & (-i u_x - u_y)\sin\left(\frac{\alpha}{2}\right) \\ (-i u_x + u_y)\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) + i u_z \sin\left(\frac{\alpha}{2}\right) \end{bmatrix}$$

with the basis $|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$^{(s)}\hat{R}_u(\alpha) \equiv \begin{bmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{bmatrix}$$

Rotation through $\alpha = 2\pi$

$$^{(v)}R_u(2\pi) = \mathbb{I}$$

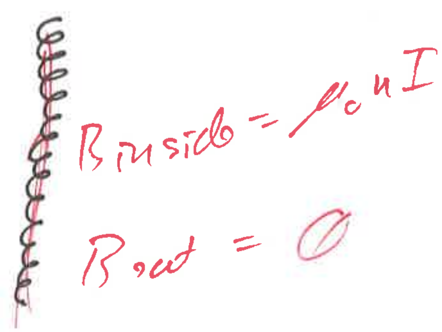
$\boxed{s=\frac{1}{2}}$ $\quad ^{(s)}R_u(2\pi) = \cos(\pi)\,\mathbb{I} = -\mathbb{I} \neq {}^{(s)}R_u(0) = \mathbb{I}$

Lie algebra is still $so(3) = su(2)$ but the groups

$$SU(2) \neq SO(3)$$

$SU(2)$ is the double cover of $SO(3)$.

Can't detect a global phase, but we can detect a relative phase. Aharonov–Bohm effect



Suppose the spinor is written in the $\{|+\rangle_z, |-\rangle_z\}$ basis in $\mathcal{E}_s$ and coordinate basis in $\mathcal{E}_r$.

$$|\psi\rangle = {}^{(r)}|\varphi\rangle \otimes {}^{(s)}|\chi\rangle \qquad \langle r|\varphi\rangle = \varphi(r)$$

$$|\psi'\rangle = \hat{R}_u(\alpha)|\psi\rangle$$

mixes spin "up" and spin "down" along $\hat{u}$

$$\begin{pmatrix} \psi'_+(\vec{r}) \\ \psi'_-(\vec{r}) \end{pmatrix} = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix} \begin{pmatrix} \psi_+(\mathcal{R}^{-1}\vec{r}) \\ \psi_-(\mathcal{R}^{-1}\vec{r}) \end{pmatrix}$$

Quaternions

one of the 4 division algebras, along with reals, complex, octonions.

$$q = a\mathbb{1} + bI + cJ + dK$$

associative, but not commutative

$$I^2 = -1 = J^2 = K^2$$

$$IJ = K, \quad JI = -K$$

$$\begin{array}{ccc} & I & \\ \nearrow & & \searrow \\ K & \leftarrow & J \end{array}$$

$$1 \to \mathbb{I}_{2\times 2}, \quad I = -i\sigma_1, \quad J = -i\sigma_2, \quad K = -i\sigma_3$$

Functions of operators
(Functions of Matrices)

used in QFT Fadeev + Popov

e.g. $e^{\underline{\underline{M}}}$ where $\underline{\underline{M}} = \begin{pmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & & \\ \vdots & \ddots & m_{nn} \end{pmatrix}$ n×n square

$$e^{\underline{\underline{M}}} \neq \begin{pmatrix} e^{m_{11}} & e^{m_{12}} & \cdots \\ \vdots & \ddots & \\ & & e^{m_{nn}} \end{pmatrix}$$

$$e^{\underline{\underline{M}}} = \mathbb{I} + \underline{\underline{M}} + \frac{1}{2!}\underline{\underline{M}}\,\underline{\underline{M}} + \frac{1}{3!}\underline{\underline{M}}\,\underline{\underline{M}}\,\underline{\underline{M}} + \cdots$$

If $\underline{\underline{M}}$ is Hermitian, then we can use the spectral decomposition ($\underline{\underline{M}} = \underline{\underline{M}}^{+}$ hermitian)

$$\underline{\underline{M}}\,|u_n\rangle = a_n|u_n\rangle = \hat{M}|u_n\rangle$$

If $a_n \neq a_m$ then $\langle u_n|u_m\rangle = \emptyset$

If a_n is degenerate $\quad \underline{\underline{M}}|u_{n1}\rangle = a_n|u_{n1}\rangle$
$\quad \underline{\underline{M}}\,|u_{n2}\rangle = a_n|u_{n2}\rangle$

Choose $\langle u_{n1}|u_{n2}\rangle = 0$ in the eigensubspace.

and normalize $\langle u_n|u_j\rangle = \delta_{nj}$

$$\underline{\underline{M}} = \sum_k a_k |u_k\rangle\langle u_k| = \sum_k a_k \underline{\underline{P_k}}$$

$P_k \equiv |u_k\rangle\langle u_k|$ is the projector onto the k^{th} eigen sub space

write suggestively: $\underline{\underline{M}}' = \sum_k a_k' \underline{\underline{P_k}}$

$\underline{\underline{M}}^\circ = \sum_k a_k^\circ P_k$

$\underline{\underline{I}} = \sum_k |u_k\rangle\langle u_k| = \sum_k \underline{\underline{P_k}}$ closure, completeness

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Projectors are idempotent $\underline{\underline{P_k}}\,\underline{\underline{P_k}} = \underline{\underline{P_k}} \Rightarrow \underline{\underline{P_k}}^n = \underline{\underline{P_k}}$

and orthogonal $\underline{\underline{P_k}}\,\underline{\underline{P_j}} = \underline{\underline{0}}$
$k \neq j$

$$\underline{\underline{P_k}}\,\underline{\underline{P_j}} = \delta_{kj} \underline{\underline{P_k}} = \delta_{kj} \underline{\underline{P_j}}$$

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what is $\underline{\underline{M}}^2 = \underline{\underline{M}}\,\underline{\underline{M}}$

$$= \sum_k a_k |u_k\rangle\langle u_k| \sum_j a_j |u_j\rangle\langle u_j| = \sum_{kj} a_k a_j |u_k\rangle \underbrace{\langle u_k|u_j\rangle}_{\delta_{kj}} \langle u_j|$$

$$= \sum_j a_j^2 |u_j\rangle\langle u_j| = \sum_k a_j^2 \underline{\underline{P_j}}$$

$$\underline{\underline{M}}^n = \sum_k a_k^n |u_k\rangle\langle u_k| = \sum_k a_k^n \underline{\underline{P_k}}$$