$f(\underline{\underline{M}}) = ?$ If f has a Taylor expansion

$$f(x) = \sum_{r=0}^{\infty} c_r x^r = c_0 + c_1 r + c_2 r^2 + \cdots$$

$$f(\underline{\underline{M}}) = \sum_{r=0}^{\infty} c_r \underline{\underline{M}}^r = \sum_{r=0}^{\infty} c_r \sum_{n=} a_n^r \underline{\underline{P}}_n$$

$$= \sum_n \left(\sum_{r=0}^{\infty} c_r a_n^r \right) \underline{\underline{P}}_n = \sum_n f(a_n) \underline{\underline{P}}_n$$

eg. $e^{\underline{\underline{M}}} = \sum_n e^{a_n} \underline{\underline{P}}_n$

$$\underline{\underline{M}} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} = M^\dagger \Longrightarrow \begin{array}{ll} a_1 = 4 & \hat{u}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ a_2 = 9 & \hat{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{array}$$

$$\underline{\underline{P}}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}}(1, -2) = \begin{pmatrix} 1/5 & -\frac{2}{5} \\ -2/5 & 4/5 \end{pmatrix}$$

$$\underline{\underline{P}}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \frac{1}{5}(2, 1) = \begin{pmatrix} \frac{4}{5} & 2/5 \\ 2/5 & 1/5 \end{pmatrix}$$

$$\underline{\underline{P}}_i = \hat{u}_1 u_1^\dagger = |u_1\rangle\langle u_1|$$

$$\underline{\underline{P}}_1 \underline{\underline{P}}_1 = \underline{\underline{P}}_1 \checkmark \quad , \quad \underline{\underline{P}}_2 \underline{\underline{P}}_2 = \underline{\underline{P}}_2 \checkmark \quad , \quad \underline{\underline{P}}_1 \underline{\underline{P}}_2 = \underline{\underline{0}} = \underline{\underline{P}}_2 \underline{\underline{P}}_1$$

$$\underline{\underline{P}}_1 + \underline{\underline{P}}_2 = \mathbb{I} \checkmark$$

$$\underline{\underline{M}} = \sum_n a_n \underline{\underline{P}}_n = 4\underline{\underline{P}}_1 + 9\underline{\underline{P}}_2 = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \checkmark$$

$$R = \sqrt{\underline{\underline{M}}} = \sum_n \sqrt{a_n}\, \underline{\underline{P}}_n = \pm 2 \underline{\underline{P}}_1 \pm 3 \underline{\underline{P}}_2$$

even though $\sqrt{x}$ does not have a taylor expansion

signs are independent

4 roots

$$\underline{\underline{R}}_i^2 = \underline{\underline{M}} \Rightarrow \underline{\underline{R}}_1, \underline{\underline{R}}_2, \underline{\underline{R}}_3, \underline{\underline{R}}_4$$

$$x^{\frac{1}{2}} \neq \sum_{n=0}^{\infty} c_n x^n$$

$$e^{\underline{\underline{M}}} = \sum_n e^{a_n} \underline{\underline{P}}_n = e^4 \underline{\underline{P}}_1 + e^9 \underline{\underline{P}}_2$$

$$\underline{\underline{M}}^{-1} = \sum_n \left(\frac{1}{a_n}\right) \underline{\underline{P}}_n = \frac{1}{4}\underline{\underline{P}}_1 + \frac{1}{9}\underline{\underline{P}}_2 \quad \Big| \quad \underline{\underline{M}}^{-1}\underline{\underline{M}} = \mathbb{I}$$

Helicity Operator

$$\hat{h} \equiv \hat{\vec{S}} \cdot \hat{\vec{p}} = \text{spin in the direction of particle momentum.}$$

spin op $\uparrow$ momentum op.

If $m \neq 0$ and $v < c$, you can outrun the particle

$\Rightarrow$ change the sign of $\vec{p}$ $\Rightarrow$ change the sign of $\hat{h}$.

For massless particles, right helicity $\Leftrightarrow$ right chirality

left helicity $\Leftrightarrow$ left chirality.

chiral $P_L = \frac{\mathbb{I} - \gamma_5}{2}$ $\quad P_R = \frac{\mathbb{I} + \gamma_5}{2}$ $\quad \underbrace{\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3}_{4\times4 \text{ matrices}}$

Addition of Angular Momenta

We saw for one spin: $|\psi\rangle = |\text{orbital}\rangle \otimes |\text{spin}\rangle$ ← tensor product

$\mathcal{E}^r$ $\quad$ $\mathcal{E}^s$

Now with two spin: $|\psi\rangle = |\text{orbital}\rangle \otimes |\text{spin}_1\rangle \otimes |\text{spin}_2\rangle$

tensor products

$\mathcal{E}^r$ $\quad$ $\mathcal{E}^{s1}$ $\quad$ $\mathcal{E}^{s2}$

everything in $\mathcal{E}^r$ commutes with everything in $\mathcal{E}^{s1}, \mathcal{E}^{s2}$, and everything $\mathcal{E}^{s1}$ commutes with everything in $\mathcal{E}^{s2}$.

Describe the spin part of the ket, eg. $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$.

$$|+\rangle_1 \otimes |+\rangle_2 \equiv |++\rangle, \quad |+\rangle_1 \otimes |-\rangle_2 \equiv |+-\rangle \text{ etc.}$$

Really $|\hat{S}_1^2, \hat{S}_2^2, \hat{S}_{1z}, \hat{S}_{2z}\rangle = |s_1, s_2, m_{s1}, m_{s2}\rangle$

One basis could be $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$ along z direct.

$$\hat{S}_1^2 |\pm\pm\rangle = \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_1 |\pm\pm\rangle = s_1(s_1+1)\hbar^2 |\pm\pm\rangle = \frac{3}{4}\hbar^2 |\pm\pm\rangle$$

$$\hat{S}_2^2 |\pm\pm\rangle = s_2(s_2+1)\hbar^2 |\pm\pm\rangle = \frac{3}{4}\hbar^2 |\pm\pm\rangle$$

$$\hat{S}_{1z} |+\pm\rangle = \frac{\hbar}{2} |+\pm\rangle = m_{s_1} |+\pm\rangle, \quad \hat{S}_{1z} |-\pm\rangle = -\frac{\hbar}{2} |-\pm\rangle$$

$$\hat{S}_{2z} |\pm +\rangle = \frac{\hbar}{2} |\pm +\rangle, \quad \hat{S}_{2z} |\pm -\rangle = -\frac{\hbar}{2} |\pm -\rangle$$

Another Basis

Define Total Spin $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 = \begin{cases} \hat{S}_x = \hat{S}_{1x} + \hat{S}_{2x} \\ \hat{S}_y = \hat{S}_{1y} + \hat{S}_{2y} \\ S_z = \hat{S}_{1z} + \hat{S}_{2z} \end{cases}$

Is $\vec{S}$ an angular momentum? Yes! Check Commutators

$$[\hat{S}_x, \hat{S}_y] = [(\hat{S}_{1x} + \hat{S}_{2x}), (\hat{S}_{1y} + \hat{S}_{2y})]$$

$$= [\hat{S}_{1x}, \hat{S}_{1y}] + \cancel{[\hat{S}_{1x}, \hat{S}_{2y}]}^{0} + \cancel{[\hat{S}_{2x}, \hat{S}_{1y}]}^{0} + [\hat{S}_{2x}, \hat{S}_{2y}]$$

$$= i\hbar \hat{S}_{1z} + i\hbar \hat{S}_{2z} = i\hbar(\hat{S}_{1z} + \hat{S}_{2z}) = i\hbar \hat{S}_z \checkmark$$

Total Spin Squared

$$\hat{S}^2 = \hat{\vec{S}} \cdot \hat{\vec{S}} = (\hat{\vec{S}}_1 + \hat{\vec{S}}_2) \cdot (\hat{\vec{S}}_1 + \hat{\vec{S}}_2)$$

$$= \hat{S}_1^2 + \hat{S}_2^2 + \hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 + \hat{\vec{S}}_2 \cdot \hat{\vec{S}}_1 =$$

tensor product, so commutator = $\emptyset$

$$= \hat{S}_1^2 + \hat{S}_2^2 + 2\,\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2$$

$$\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 = \hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \hat{S}_{1z}\hat{S}_{2z}$$

$$= \frac{1}{2}\left(\hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right) + \hat{S}_{1z}\hat{S}_{2z}$$

$\hat{S}_{1\pm} = \hat{S}_{1x} \pm i\hat{S}_{1y}$

$\hat{S}_{2\pm} = \hat{S}_{2x} \pm i\hat{S}_{2y}$

Other Basis $\left\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\right\}$ See if this is a CSCO

$[\hat{S}_1^2, \hat{S}_2^2] = 0$ tensor product ✓

$$[\hat{S}_z, \hat{S}_1^2] = [(\hat{S}_{1z} + \hat{S}_{2z}), \hat{S}_1^2] = \cancelto{0}{[\hat{S}_{1z}, \hat{S}_1^2]} + \cancelto{0}{[\hat{S}_{2z}, \hat{S}_1^2]} = 0$$ ✓

$[\hat{\vec{S}}_1, \hat{S}_1^2] = 0$

$[\hat{S}_z, \hat{S}_2^2] = 0$ ✓

$$[\hat{S}^2, \hat{S}_1^2] = [(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2), \hat{S}_1^2] = 0$$ ✓

$[\hat{\vec{S}}_1, \hat{S}_1^2] = 0$

$[\hat{S}^2, \hat{S}_2^2] = 0$ ✓

$$[\hat{S}^2, \hat{S}_{1z}] = [(\cancel{\hat{S}_1^2 + \hat{S}_2^2} + 2\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2), \hat{S}_{1z}]$$

$$= 2[\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2, \hat{S}_{1z}] = 2[(\hat{S}_{1x}\hat{S}_{2x} + \hat{S}_{1y}\hat{S}_{2y} + \cancel{\hat{S}_{1z}\hat{S}_{2z}}), \hat{S}_{1z}]$$

$$= 2[\hat{S}_{1x}, \hat{S}_{1z}]\hat{S}_{2x} + 2[\hat{S}_{1y}, \hat{S}_{1z}]\hat{S}_{2y}$$

$$= 2i\hbar(-\hat{S}_{1y}\hat{S}_{2x} + \hat{S}_{1x}\hat{S}_{2y}) \neq 0$$

$$[\hat{S}^2, \hat{S}_{2z}] = \underset{1 \Leftrightarrow 2}{\text{switch}} = -[\hat{S}^2, \hat{S}_{1z}]$$

$$[\hat{S}^2, \hat{S}_z] = [\hat{S}^2, (\hat{S}_{1z} + \hat{S}_{2z})] = 0 \checkmark$$

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set # 1 CSCO $\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}_{1z}, \hat{S}_{2z}\}$

set # 2 CSCO $\{\hat{S}_1^2, \hat{S}_2^2, \hat{S}^2, \hat{S}_z\}$