$\hat{\vec{S}}$ is an angular momentum $\Rightarrow$ S is integer or oddhalf integer ≥ 0

M (quantum number associated with $\hat{S}_z$) jumps by one unit from $-S, -S+1, \dots S-1, S$

Express states in the new basis

$$\hat{S}_z|++\rangle = (\hat{S}_{1z} + \hat{S}_{2z})|++\rangle$$

$$= \frac{\hbar}{2}|++\rangle + \frac{\hbar}{2}|++\rangle = 1\hbar|++\rangle = M\hbar|++\rangle$$

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$$\hat{S}_z|--\rangle = (\hat{S}_{1z} + \hat{S}_{2z})|--\rangle = -1\hbar|--\rangle$$

$M=-1$

$$\hat{S}_z|+-\rangle = (\hat{S}_{1z} + \hat{S}_{2z})|+-\rangle = \frac{\hbar}{2}|+-\rangle - \frac{\hbar}{2}|+-\rangle =$$

$$= 0\,|+-\rangle$$

$M=0$

$$\hat{S}_z|-+\rangle = \dots \quad 0\,|-+\rangle$$

$M=0$

$$\hat{S}^2 = \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}$$

$$\hat{S}^2|++\rangle = \tfrac{3}{4}\hbar^2|++\rangle + \tfrac{3}{4}\hbar^2|++\rangle + 2\left(\tfrac{\hbar}{2}\right)\left(\tfrac{\hbar}{2}\right)|++\rangle + 0 + 0$$
$$= 2\hbar^2|++\rangle = (s)(s+1)\hbar^2|++\rangle \implies s = 1$$

$$\hat{S}^2|--\rangle = \tfrac{3}{4}\hbar^2|--\rangle + \tfrac{3}{4}\hbar^2|--\rangle + 2\left(-\tfrac{\hbar}{2}\right)\left(-\tfrac{\hbar}{2}\right) + 0 + 0$$
$$= 2\hbar^2|--\rangle = s(s+1)\hbar^2|--\rangle \implies s = 1$$

$$\hat{S}_{1+}|-\pm\rangle = \hbar\sqrt{s_1(s_1+1) - m_1(m_1+1)}\,|+\pm\rangle$$
$$= \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \left(-\tfrac{1}{2}\right)\left(-\tfrac{1}{2}+1\right)}\,|+\pm\rangle = 1\hbar\,|+\pm\rangle$$

$$\hat{S}_{1-}|+\pm\rangle = \hbar\sqrt{s_1(s_1+1) - m_1(m_1-1)}\,|-\pm\rangle$$
$$= \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}-1\right)}\,|-\pm\rangle = 1\hbar\,|-\pm\rangle$$

$$\hat{S}^2|+-\rangle = \left(\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_{1z}\hat{S}_{2z} + \hat{S}_{1+}\hat{S}_{2-} + \hat{S}_{1-}\hat{S}_{2+}\right)|+-\rangle$$
$$= \tfrac{3}{4}\hbar^2|+-\rangle + \tfrac{3}{4}\hbar^2|+-\rangle + 2\left(\tfrac{\hbar}{2}\right)\left(-\tfrac{\hbar}{2}\right)|+-\rangle + 0$$
$$+ \hbar^2\,|-+\rangle$$

$$= \hbar^2|+-\rangle + \hbar^2|-+\rangle$$

$$\hat{S}^2|-+\rangle = \tfrac{3}{4}\hbar^2|-+\rangle + \tfrac{3}{4}\hbar^2|-+\rangle + 2\left(-\tfrac{\hbar}{2}\right)\left(\tfrac{\hbar}{2}\right)|-+\rangle$$
$$+\hbar^2|+-\rangle + \emptyset$$
$$= \hbar^2|-+\rangle + \hbar^2|+-\rangle$$

$$\hat{S}^2 \cdot \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) = 2\hbar^2 \tfrac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$
$$\uparrow s(s+1) \Rightarrow s = 1$$

$$\hat{S}^2 \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = \emptyset\ \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$
$$\uparrow s(s+1) \Rightarrow s = 0$$

$$|S=1, M=0\rangle = \tfrac{1}{\sqrt{2}}\left(|\underset{m_1}{+}\underset{m_2}{-}\rangle + |\underset{m_1}{-}\underset{m_2}{+}\rangle\right)$$

$$|S=1, M=1\rangle = |\underset{m_1}{+}\underset{m_2}{+}\rangle$$

$$|S=1, M=-1\rangle = |--\rangle$$

triplet
symmetric under
$1 \rightleftarrows 2$ interchange

$$|S=0, M=0\rangle = \tfrac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

singlet
antisymmetric

Orthonormality

$\langle 11|11\rangle = 1$, $\langle 11|00\rangle = 0$ etc.

Add more than 2 angular momenta

e.g. $\hat{\vec{L}}_1$, $\hat{\vec{L}}_2$, $\hat{\vec{S}}_1$ in any order pairwise

$\Rightarrow (\hat{\vec{L}}_1 + \hat{\vec{L}}_2) + \hat{\vec{S}}_1$ or $(\hat{\vec{S}}_1 + \hat{\vec{L}}_2) + \hat{\vec{L}}_1$

Now add two angular momenta $\hat{\vec{J}}_1$ and $\hat{\vec{J}}_2$

$\hat{J}_{1z}$ has eigenvalues $m_1 \in \underbrace{\{-j_1, -j_1+1, \ldots, j_1-1, j_1\}}_{(2j_1+1) \text{ of these}}$

$\hat{J}_1^2 |j_1, m_1\rangle = j_1(j_1+1)\hbar^2 |j_1, m_1\rangle$

$\hat{J}_{2z}$ " " $m_2 \in \underbrace{\{-j_2 \; --- \; j_2\}}_{(2j_2+1) \text{ of these}}$

The number of the kets in $\mathcal{E}^1 \otimes \mathcal{E}^2 = (2j_1+1)(2j_2+1)$

Now we want to change bases from

$\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}_{1z}, \hat{J}_{2z}\}$ to $\{\hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z\}$

$|j_1, j_2, m_1, m_2\rangle \equiv |m_1, m_2\rangle$ | $|j_1, j_2, J, M\rangle \equiv |J, M\rangle$

$$\hat{\vec{J}} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2 \quad \text{s} \quad \hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z}$$

Because $\hat{\vec{J}}$ is an angular momentum

$$M \in \underbrace{\{-J, -J+1, \dots, J-1, J\}}_{(2J+1) \text{ of these}}$$

The maximum value of M is $(j_1 + j_2) \Rightarrow J_{max} = j_1 + j_2$

What is J_{min}?

# of kets must be the same, no matter the basis

$$\sum_{J=J_{min}}^{J_{max} = j_1+j_2} (2J+1) = (2j_1+1)(2j_2+1) \Rightarrow J_{min} = |j_1 - j_2|$$

Proof: Assume $j_1 \geq j_2$

$$\sum_{J=j_1-j_2}^{j_1+j_2} (2J+1)$$

J is dummy index. change to $K = J - j_1 + j_2$ $\Rightarrow J = K + j_1 - j_2$

$$= \sum_{K=0}^{2j_2} [2(K + j_1 - j_2) + 1]$$

$$= 2\underbrace{\sum_{k=0}^{2j_2} k}_{2\frac{(2j_2)(2j_2+1)}{2}} + \underbrace{\sum_{k=0}^{2j_2}(2j_1 - 2j_2 + 1)}_{(2j_1-2j_2+1)(2j_2+1)}$$

$$= (2j_1+1)(2j_2+1) \checkmark$$

The ket with the largest values of J and M is

$$\underbrace{|J=j_1+j_2, M=j_1+j_2\rangle}_{\text{CSCO } \hat{J}_1^2, \hat{J}_2^2, \hat{J}^2, \hat{J}_z} = |m_1=j_1\rangle \otimes |m_2=j_2\rangle \equiv \underbrace{|j_1, j_2\rangle}_{\text{CSCO is } \hat{J}_1^2 \hat{J}_2^2 \hat{J}_{1z} \hat{J}_{2z}}$$

This ket is unique (degeneracy 1).

Act with the lowering operator: $\hat{J}_- |J,M\rangle$

$$= \hbar\sqrt{J(J+1) - M(M-1)}\,|J, M-1\rangle$$

continue until you reach $|J,-J\rangle$.

This will fix the otherwise arbitrary phases.

This ket is known $|J, J-1\rangle$

But another ket $|J-1, J-1\rangle$ has the same M,

$$\hat{J}_z |J, J-1\rangle = \hbar (J-1) |J, J-1\rangle$$
$$\hat{J}_z |J-1, J-1\rangle = \hbar (J-1) |J-1, J-1\rangle$$

But the two kets must be orthogonal

$$\langle J, J-1 | J-1, J-1\rangle = \emptyset.$$

That is $|J, J-1\rangle$ is one linear combination of

$|J_1, J_1-1\rangle \otimes |J_2, J_2\rangle$ and $|J_1, J_1\rangle \otimes |J_2, J_2-1\rangle$

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while $|J-1, J-1\rangle$ is the other linear combination.

E.g. $|J=1, M=1\rangle = |++\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2$

$$J_-|11\rangle = \hbar\sqrt{J(J+1)-M(M-1)}\,|10\rangle$$

$$= \hbar\sqrt{1(1+1)-1(1-1)} = \sqrt{2}\hbar\,|10\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}\hbar} J_-|++\rangle = (J_{1-}+J_{2-})|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2$$

$$= \frac{1}{\sqrt{2}\hbar}\Big[\hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2$$

$$+\ \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2\Big]$$

$$= \frac{1}{\sqrt{2}}\big[|-+\rangle + |+-\rangle\big] \checkmark$$

One more time

$$J_-|1,0\rangle = \hbar\sqrt{1(1+1)-0(0-1)}\,|1,-1\rangle = \sqrt{2}\hbar\,|1,-1\rangle$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}\hbar}(J_{1-}+J_{2-})\frac{1}{\sqrt{2}}\big[|-+\rangle + |+-\rangle\big]$$

$$= \frac{1}{2}\big[|--\rangle + |--\rangle\big] = |--\rangle$$

$|0,0\rangle$ is the orthogonal combination of the kets in $|1,0\rangle$

$|0,0\rangle = \frac{1}{\sqrt{2}}\big[|+-\rangle - |-+\rangle\big]$ up to phase