Clebsch - Gordan Coefficients

$j_1 = 1 \quad j_2 = 1$

# of kets

$(2j_1+1)(2j_2+1)$

$3 \times 3 = \boxed{9}$

$J_{max} = j_1 + j_2 = 2 \quad (2\cdot 2+1) = 5$

also $J = 1 \quad 3$

$J_{min} = |j_1 - j_2| = 0 \quad 1$

$+9$ ✓

$$\begin{array}{ccccc} j_1 & & j_2 \\ 1 & \times & 1 \end{array} \qquad \begin{array}{cc} m_1 & m_2 \\ \hline +1 & +1 \end{array} \qquad \begin{array}{|c|} 2 \\ +2 \\ \hline 1 \end{array} \begin{array}{l} J \\ M \\ \text{C-G coefficient} \end{array}$$

$$|J=2, M=+2\rangle = 1\,|j_1=1, m_1=1\rangle \otimes |j_2=1, m_2=1\rangle$$

$$= 1\,|\underset{j_1}{1}\,\underset{j_2}{1}\,\underset{m_1}{1}\,\underset{m_2}{1}\rangle = 1\,|\underset{m_1}{1}\,\underset{m_2}{1}\rangle$$

$$\begin{array}{ccc} j_1 & & j_2 \\ 1 & * & 1 \end{array} \qquad \begin{array}{|c|} 2 \\ +1 \end{array} \begin{array}{l} J \\ M \end{array}$$

$$\begin{array}{c|cc|} m_1 m_2 & +1 & 0 \\ & 0 & +1 \end{array} \qquad \begin{array}{|c|} 1/2 \\ 1/2 \end{array}$$

$$|J=2, M=+1\rangle =$$

$$= \sqrt{\tfrac{1}{2}}\,|j_1=1, m_1=1\rangle \otimes |j_2=1, m_2=0\rangle$$

$$+ \sqrt{\tfrac{1}{2}}\,|j_1=1, m_1=0\rangle \otimes |j_2=1, m_2=1\rangle$$

$$|\underset{J}{2},\underset{M}{1}\rangle = \frac{1}{\sqrt{2}}\left[\,|\underset{m_1}{1}\,\underset{m_2}{0}\rangle + |\underset{m_1}{0}\,\underset{m_2}{1}\rangle\,\right]$$

$$\overset{j_1}{1} \times \overset{j_2}{1}$$

$$\begin{array}{|c|l} \hline 1 & - J \\ 0 & - M \\ \hline \end{array}$$

$$|J=1, M=0\rangle = \frac{1}{\sqrt{2}}\left[\,|\underset{m_1}{1},\underset{m_2}{-1}\rangle - |\underset{m_1}{-1},\underset{m_2}{1}\rangle\,\right]$$

$$\begin{array}{l} m_1 \; m_2 \end{array} \begin{array}{|cc|} \hline +1 & -1 \\ 0 & 0 \\ -1 & +1 \\ \hline \end{array} \quad \begin{array}{|c|l} \hline 1/2 & \\ 0 & \text{CG} \\ -1/2 & \\ \hline \end{array}$$

does not contain $|\underset{m_1}{0}\,\underset{m_2}{0}\rangle$

$$\overset{j_1}{1} \times \overset{j_2}{1}$$

$$\begin{array}{|ccc|l} \hline 2 & 1 & 0 & J \\ 0 & 0 & 0 & M \\ \hline \end{array}$$

$$\begin{array}{|cc|} \hline 0 & 0 \\ \hline m_1 & m_2 \end{array} \quad \begin{array}{|ccc|} \hline 2/3 & 0 & -1/3 \\ \hline \end{array}$$

$$|m_1=0, m_2=0\rangle = \sqrt{\frac{2}{3}}\,|J=2, M=0\rangle - \sqrt{\frac{1}{3}}\,|J=0, M=0\rangle$$

$|\underset{j_1}{1}\,\underset{j_2}{1}\,\underset{m_1}{0}\,\underset{m_2}{0}\rangle$

does not contain $|J=1, M=0\rangle$

Tensors e.g. electric multipole tensors

scalar: monopole = charge

$$Q = \int dV \rho(\vec{r})$$ 1 component

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vector: electric dipole

$$\vec{Q} = \int dV \rho(\vec{r}) \vec{r}$$

component form

$$Q_i = \int dV \rho(\vec{r}) X_i$$ 3 components

Cartesian tensors

rank $k=2$ tensor = quadrupole tensor

$$Q_{ij} = \int dV \rho(\vec{r}) X_i X_j$$ symmetric $i \leftrightarrow j$

x	x	x
	x	x
		x

9 components

~~6 independent~~

5 independent components

traceless

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rank $k=3$ octapole tensor traceless

$$Q_{ijk} = \int dV \rho(\vec{r}) X_i X_j X_k$$ 7 independent components

Spherical tensors

$$Q^{l}_{m} = \int dV\, \rho(\bar{r})\, r^{l}\, Y^{*}_{lm}(\theta,\varphi)\sqrt{\frac{2\pi}{2l+1}}$$

rank

scalar $Q^{0=k}_{0} = Q$ charge

$Q^{+1}_{+1} = Q_x + i Q_y$

vector $Q^{1=k}_{m} \Longleftrightarrow Q_i$

m: $+1, 0, -1$

quadrupole tensor $Q^{2=k}_{m}$

m: $+2, +1, 0, -1, -2$

5 independent components.

Wigner-Eckart Theorem

Matrix elements of spherical tensor operators (e.g. scalar, vectors, ...) can be written as a Clebsch-Gordan coefficient ~~x~~ a factor independent of orientation.

(scalar $k=0$, vector $k=1$)

rank-k

$$\langle n'l'm'|\hat{T}^{(k)}_q|nlm\rangle = \underbrace{\langle \overset{j_1 m_1}{l'm'}\,\overset{j_2 k}{kq}|\overset{m_2}{lm}\rangle}_{\text{Clebsch-Gordan coeffient}}\underbrace{\langle n'l'||\hat{T}^{(k)}||nl\rangle}_{\text{reduced matrix element}}$$

other quantum numbers $\quad \hat{L}^2 \quad \hat{L}_z \quad$ component

does not depend on m, m', or q.

operating with a spherical tensor of rank k on an angular momentum j_1 eigenstate is "like" adding j_1+k
$\Rightarrow$ Clebsch Gordan Coefficients $\Longrightarrow$ selection rules

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① WE for scalars (spherical tensor rank $0=k$) $\hat{f}$

cf. lecture #19 $\quad [\hat{\vec{L}},\hat{f}]=0 \Rightarrow [\hat{L}_i,\hat{f}]=0$ for $i=1,2,3$

$$[\hat{L}_z,\hat{f}]=0,\ [\hat{L}_\pm,\hat{f}]=0,\ [\hat{L}^2,\hat{f}]=0$$

$\hat{f}=\hat{H}$ Hamiltonian, e.g.

$$\langle n'l'm'|[\hat{L}_z,\hat{f}]|nlm\rangle = 0$$

$$\langle n'l'm'|\hat{L}_z\hat{f}|nlm\rangle - \langle n'l'm'|\hat{f}\hat{L}_z|nlm\rangle = 0$$

$$(m'-m)\,\langle n'l'm'|\hat{f}|nlm\rangle = 0$$

Matrix elements of a scalar operator $\hat{f}$ vanish unless

$\Delta m \equiv m'-m = 0$

$$\langle n'l'm'|[\hat{L}^2,\hat{f}]|nlm\rangle = 0$$

$$\langle n'l'm'|\hat{L}^2\hat{f}|nlm\rangle - \langle n'l'm'|\hat{f}\hat{L}^2|nlm\rangle = 0$$

$$[l'(l'+1)-l(l+1)]\,\langle n'l'm'|\hat{f}|nlm\rangle = 0$$

Matrix elements of a scalar operator vanish unless

$\Delta l = l'-l = 0$

$$\langle n'l'm'|[\hat{L}_+,\hat{f}]|nlm\rangle = 0$$

$$\langle n'l'm'|\hat{L}_+\hat{f}|nlm\rangle - \langle n'l'm'|\hat{f}\hat{L}_+|nlm\rangle = 0$$

$$\langle \hat{L}_-(n'l'm')|\hat{f}|nlm\rangle - \langle n'l'm'|\hat{f}|\hat{L}_+(nlm)\rangle$$

$$\hbar\sqrt{l'(l'+1)-m'(m'-1)}\,\langle n'l'(m'-1)|\hat{f}|nlm\rangle$$
$$-\hbar\sqrt{l(l+1)-m(m+1)}\,\langle n'l'm'|\hat{f}|nl(m+1)\rangle=0$$

When $\Delta m=1$, $\Delta l=0$

$$\langle n'l\,m|\hat{f}|nlm\rangle=\langle n'l(m+1)|\hat{f}|nl(m+1)\rangle$$

Matrix elements of a scalar operator are independent of m. (orientation)

$$\langle n'l'm'|\hat{f}|nlm\rangle=\delta_{ll'}\delta_{mm'}\underbrace{\langle n'l'\|f\|nl\rangle}_{\text{depends on } n,n',l,\cancel{l'} \text{ but not } m,\cancel{m'}}$$

e.g. hydrogen 2p wave function $\hat{r}^2$ is a scalar op

$$\text{WE} \Rightarrow \langle 211|\hat{r}^2|211\rangle=30a_0^2=\langle 210|\hat{r}^2|210\rangle$$
$$=\langle 21-1|\hat{r}^2|21-1\rangle$$

only need to calculate one integral.