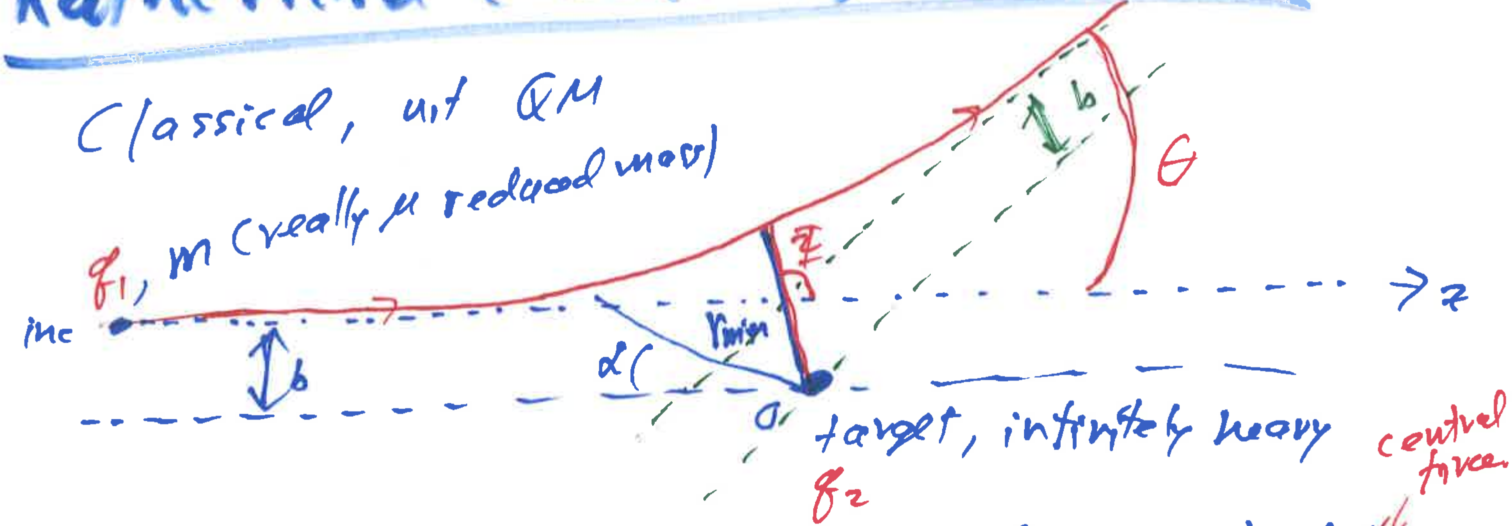
Rutherford (Coulomb) Scattering

Classical, w/o QM



Conservation of Energy $E = \frac{1}{2}m(\underbrace{\dot{r}^2 + r^2\dot{\alpha}^2}_{v_{inc}^2}) + V(r)$

$E = \frac{1}{2}mv^2$

Conservation of Angular Momentum

$L = mvb \Rightarrow v = \sqrt{\frac{2E}{m}}$ $\qquad L = mr^2\dot{\alpha} \Rightarrow \dot{\alpha} = \frac{d\alpha}{dt} = \frac{L}{mr^2}$

$$\dot{r}^2 + \frac{L^2}{m^2r^2} = \frac{2}{m}(E-V)$$

Define $u = \frac{1}{r}$

$r = \frac{1}{u} \rightarrow \frac{dr}{du} = -\frac{1}{u^2}$

$$b = \frac{L}{m}\sqrt{\frac{m}{2E}} = \frac{L}{\sqrt{2mE}}$$

$$\dot{r} = \frac{dr}{dt} = \left(\frac{dr}{du}\right)\left(\frac{du}{d\alpha}\right)\left(\frac{d\alpha}{dt}\right) = \left(\frac{-1}{\cancel{u^2}}\right)\left(\frac{du}{d\alpha}\right)\left(\frac{Lu^{\cancel{2}}}{m}\right)$$

$$\dot{r}^2 + \frac{L^2u^2}{m^2} = \frac{2}{m}(E-V)$$

$$\left(\frac{L}{m}\frac{du}{d\alpha}\right)^2 + \frac{L^2u^2}{m^2} = \frac{2}{m}(E-V)$$

$$\frac{du}{d\alpha} = \sqrt{\frac{2m}{L^2}(E-V) - u^2}$$

$$d\alpha = \frac{du}{\sqrt{\frac{2m}{L^2}\left[E - V(u)\right] - u^2}}$$

$$V = \frac{q_1 q_2}{4\pi\epsilon_0 r} = \frac{q_1 q_2 u}{4\pi\epsilon_0}$$

$$b = \frac{q_1 q_2}{8\pi\epsilon_0 E}\cot\left(\frac{\theta}{2}\right)$$

differential cross section

$$\sigma(\theta,\varphi) = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \left[\frac{q_1 q_2}{16\pi\epsilon_0 E \sin^2\left(\frac{\theta}{2}\right)}\right]^2$$

total cross section

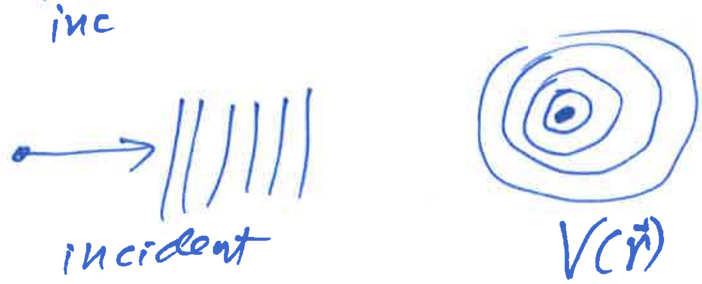
$$\sigma = \iint \underbrace{\sigma(\theta,\varphi)}_{\boxed{\frac{d\sigma}{d\Omega}}} d\Omega = \iint \sin\theta \, d\theta \, d\varphi \, \underbrace{\sigma(\theta,\varphi)}_{\boxed{\frac{d\sigma}{d\Omega}}}$$

$$\propto \int \sin\theta \, d\theta \, \csc^4\left(\frac{\theta}{2}\right) \rightarrow \infty$$

Quantum Mechanical Scattering

Assume plane waves incident

$$\psi_{inc}(\vec{r}) = A\, e^{ikz}, \quad \Psi_{inc}(\vec{r},t) = \psi_{inc}(\vec{r})\, e^{-\frac{iEt}{\hbar}}, \quad E = \frac{\hbar^2 k^2}{2m}$$



outgoing spherical waves

total wave function with scattering

$$\psi(\vec{r}) = A\left\{ e^{ikz} + f_k(\theta,\varphi)\frac{e^{ikr}}{r} \right\} \quad \text{for large } r$$

Recognize $\frac{e^{ikr}}{r}$ as the retarded Green function for the wave equation – represents outgoing spherical waves.

Advanced Green function $\frac{e^{-ikr}}{r}$ incoming spherical waves,

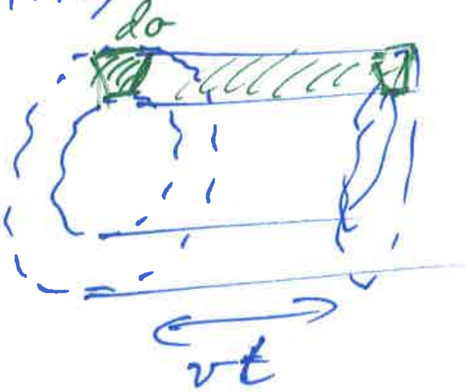
Standing Green function $\frac{\cos(kr)}{r}$ combines incoming and outgoing waves into a standing wave

$\frac{\sin(kr)}{r}$ not allowed, finite at $r=0$

$$\nabla^2(G) \propto \delta^{(3)}(r) \propto \nabla^2\left(\frac{1}{r}\right)$$

$f_k(\theta,\varphi)$ is the scattering amplitude,
depends on the potential $V(\vec{r})$,
obtained by solving the Schrödinger Equation.

Probability that the incident wave passes through the infinitesimal area $d\sigma$



$$dP = |\psi_{inc}|^2 \underbrace{dV}_{} = |A|^2 \underbrace{(v\,dt)\,d\sigma}_{}$$

equals the probability that the wave scatters into solid angle $d\Omega$

$$dP = |\psi_{scat}|^2 dV = \frac{|A|^2 |f|^2}{r^2}(v\,dt)\,d\Omega$$

$$d\sigma = |f|^2 d\Omega \Rightarrow \frac{d\sigma}{d\Omega} = \sigma(\theta,\varphi) = |f_k(\theta,\varphi)|^2$$

the differential scattering cross section is the absolute square of the scattering amplitude.

Partial Wave Analysis

Remember $\psi(r,\theta,\varphi) = R(r)\,Y_{lm}(\theta,\varphi)$, $\quad u(r) = R(r)\cdot r$

$$\text{TISE} \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u(r) = E\,u(r)$$

Assume $V(r) \to 0$ faster than $\frac{1}{r^2}$ outside some finite radius (V is localized). That's why we can't analyze Rutherford scattering with partial waves.

for large r $\quad \dfrac{d^2u}{dr} - \dfrac{l(l+1)}{r^2}u(r) = -k^2u(r)$

$E = \dfrac{\hbar k^2}{2m}$

Solutions are $u(r) = C r\, j_l(kr) + D r\, n_l(kr)$

spherical Bessel functions

But we need linear combinations

$$h_l^{(1)}(kr) = j_l(kr) + i n_l(kr), \quad h_l^{(2)}(kr) = j_l(kr) - i n_l(kr)$$

spherical Hankel functions

outgoing
$$h_0^{(1)}(kr) = -i\frac{e^{ikr}}{kr} \qquad h_0^{(2)}(kr) = +i\frac{e^{-ikr}}{kr}$$

incoming

$$h_1^{(1)}(kr) = \left[\frac{-i}{(kr)^2} - \frac{1}{kr}\right]e^{ikr}$$

$$\psi(r,\theta,\varphi) = A\left\{ e^{ik\overset{r\cos\theta}{z}} + \sum_{l,m} F_{lm}\, h_l^{(1)}(kr)\, Y_{lm}(\theta,\varphi) \right\}$$

$$m \in \underbrace{\{-l, -l+1 \ldots\ l-1, l\}}_{2l+1}$$

Assume azimuthal symmetry (no φ dependence)

$$\underset{\uparrow m=0}{F_{l0}} \equiv i^{l+1} k \sqrt{4\pi(2l+1)}\; a_l \leftarrow \text{defines } a_l$$

$$\underset{\uparrow \text{no } \varphi}{\psi(r,\theta)} = A\left\{ e^{ikz} + k\sum_{l=0}^{\infty} i^{l+1}(2l+1)\, a_l\, h_l^{(1)}(kr)\, P_l(\cos\theta) \right\}$$

look at large r $\quad \psi(r,\theta) \to A\left\{ e^{ikz} + f_k(\theta)\frac{e^{ikr}}{r} \right\}$

$$f_k(\theta) = \sum_{l=0}^{\infty} (2l+1)\, \underline{a_l}\, P_l(\cos\theta)$$

a_l is the l^{th} partial wave amplitude.

$$\frac{d\sigma}{d\Omega} = \sigma(\theta,\varphi) = |f_k(\theta)|^2 = \sum_{l=0}^{\infty}\sum_{l'=0}^{\infty} (2l+1)(2l'+1)\, a_l^* a_{l'}\, P_l(\cos\theta)\, P_{l'}(\cos\theta)$$

$$\sigma = \iint d\Omega\, \sigma(\theta,\phi) = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)|a_\ell|^2$$

used orthogonality of $P_\ell(\cos\theta)$ Legendre polynomials

$$\int_{-1}^{+1} P_\ell(u) P_{\ell'}(u)\, du = \left(\frac{2}{2\ell+1}\right)\delta_{\ell\ell'}$$

$u = \cos\theta$, $du = -\sin\theta\, d\theta$ $\quad u=-1 \Rightarrow \theta=\pi$ $\quad u=+1 \Rightarrow \theta=0$

Expand incident wave (plane) in spherical waves

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1) j_\ell(kr) P_\ell(\cos\theta)$$

Rayleigh's formula

$h_\ell(kr)$ blows up at $r=0$, but e^{ikz} does not

$$\psi(r,\theta) = A \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\left[j_\ell(kr) + ik a_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta)$$