The SO(4) Symmetry of the Hydrogen Atom

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Abstract

We review the "hidden" SO(4) symmetry of the bound hydrogen atom. We first take an algebraic approach using the quantum mechanical analogue of the Laplace-Runge-Lenz vector in the classical Kepler problem. We then take an analytical approach by applying a stereographic projection to the momentum-space wavefunction of a hydrogen atom solution.

Introduction

This paper studies the simplest bound state hydrogen atom problem. Consider the Hilbert space $\mathcal{H} = L^2\left(\mathbf{R}^3\right)$[1]. We ask if there exists $E < 0$ and $\psi \in \mathcal{H}$ such that

$$\frac{-\hbar^2}{2m_e}\left(\nabla^2\psi\right)(\mathbf{r}) - \frac{k}{r}\psi(\mathbf{r}) = E\psi(\mathbf{r}) \tag{1}$$

where $k = \frac{e^2}{4\pi\varepsilon_0}$, $m_e$is the mass[2] of an electron, and $r = |\mathbf{r}|$. The solutions to this problem are of course known exactly. We shall demonstrate an alternative approach to this problem that was apparently originally developed by Pauli and Fock [6, 5]. Aside from being "elegant," this new approach gives insight into a so called "hidden symmetry" of the hydrogen atom.

In short, we will do the following. We will first consider the quantum mechanical analogue of the Laplace-Runge-Lenz vector (see below). Using this operator along with usual orbital angular momentum operators, we will determine the allowed energies of bound hydrogen states without solving equation 1. We will remark that angular momentum operators and Laplace-Runge-Lenz operators "generate" the Lie group $SO(4)$ when we are considering bound states. The appearance of $SO(4)$ will finally be explained as follows. When we properly apply a stereographic projection of a momentum space hydrogen wavefunction onto the 3-sphere S^3 (see below), the projected wavefunction satisfies the Schrödinger equation of a "free particle on S^3."

[1]Roughly speaking, $L^2\left(\mathbf{R}^3\right)$ is the set of measurable functions $\psi : \mathbf{R}^3 \to \mathbf{C}$ such that $\int_{\mathbf{R}^3} |\psi|^2$ converges.

[2]Throughout the entirety of this paper, it is acceptable to replace mass by a reduced mass to handle a two-body problem.

1 The Algebraic Approach[3]

The classical LRL vector

The reader already familiar with the Laplace-Runge-Lenz vector may skip this section. Consider the classical Kepler problem (the classical analogue of the hydrogen atom). Fix a three-dimensional Cartesian coordinate system. A particle of mass m in three dimensions subject to a potential $V(\mathbf{r}) = \frac{-k}{r}$ for some positive constant k. The manifold of states for the particle is now $\mathbf{R}^3 \times \mathbf{R}^3$ where the first $\mathbf{R}^3$ gives the position $\mathbf{r}$ of the particle and the second $\mathbf{R}^3$ gives the momentum $\mathbf{p}$ of the particle. We now define a vector field $\mathbf{A} : \mathbf{R}^3 \times \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ on the manifold of states by

$$\mathbf{A}(\mathbf{r}, \mathbf{p}) = \frac{1}{m}\mathbf{p} \times \mathbf{L} - k\frac{\mathbf{r}}{r}$$

where $\mathbf{L}$ denotes $\mathbf{r} \times \mathbf{p}$. This function is called the Laplace-Runge-Lenz vector[4].

The most important fact about $\mathbf{A}$ is that it is a conserved quantity. That is, if $\mathbf{r}(t)$ is a curve in $\mathbf{R}^3$ satisfying Newton's second law $m\ddot{\mathbf{r}} = \frac{-k}{r^2}\hat{\mathbf{r}}$, then $\mathbf{A}\left(\mathbf{r}(t), m\dot{\mathbf{r}}(t)\right)$ is a constant function of time. The following proof is given by [3]:

$$\begin{aligned}
\dot{\mathbf{A}} &= \frac{1}{m}\dot{\mathbf{p}} \times \mathbf{L} - k\frac{\dot{\mathbf{r}}}{r} + k\frac{\mathbf{r}}{r^2}\dot{r} \\
&= \frac{-k}{m}\frac{\mathbf{r}}{r^3} \times (\mathbf{r} \times \mathbf{p}) - \frac{k}{m}\frac{\mathbf{p}}{r} + k\frac{\mathbf{r}}{r^2}\dot{r} \\
&= \frac{-k}{mr^3}\left((\mathbf{r} \cdot \mathbf{p})\mathbf{r} - r^2\mathbf{p}\right) - \frac{k}{m}\frac{\mathbf{p}}{r} + k\frac{\mathbf{r}}{r^2}\dot{r} \\
&= \frac{-k\mathbf{r}}{2r^3}\frac{d}{dt}\left(r^2\right) + \frac{k\mathbf{r}}{r^2}\dot{r} \\
&= 0.
\end{aligned}$$

(The only nontrivial parts of this proof are the use of conservation of angular momentum in the first line, the use of Newton's second law in the second line, and a cross product manipulation in the third.)

One can think of the LRL vector as follows. In an elliptical orbit, $\mathbf{A}$ constantly "points" from the origin (the sun, perhaps) toward the periapsis. This interpretation fails for circular orbits where $\mathbf{A}$ is equal to 0. A similar interpretation is valid for orbits which are not bound.

The quantum mechanical LRL vector and the energy levels of hydrogen

Consider the hydrogen atom described in the introduction. Motivated by the conservation of the classical LRL vector, we define the quantum LRL vector as

[3] Our discussion closely follows those of Mahajan [3] and of Bander and Itzykson [1].

[4] Two remarks should be made. First, note that the LRL vector is only defined for the Kepler problem. Second, many authors define this vector as m multiplied by our definition.

the Hermitian operator

$$\mathbf{A} = \frac{1}{2m_e}\left(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}\right) - k\frac{\mathbf{r}}{r}$$

where $\mathbf{p}$, $\mathbf{r}$, and $\mathbf{L}$ denote the Hermitian operators for momentum, position, and angular momentum respectively. Note that the "anti-symmetrization" in this definition makes $\mathbf{A}$ Hermitian. $\mathbf{A}$ certainly reduces to the classical LRL vector in the "classical limit". We will see shortly that this operator plays a very important role in the hydrogen atom.

Let H be the Hamiltonian for the system: $H = \frac{\mathbf{p}^2}{2m_e} - \frac{k}{r}$. We claim that the following commutation relations hold.

$$\begin{aligned}
[H, L_i] &= 0 \\
[H, A_i] &= 0 \\
[L_i, L_j] &= i\hbar\varepsilon_{ijk}L_k \\
[L_i, A_j] &= i\hbar\varepsilon_{ijk}A_k \\
[A_i, A_j] &= -i\hbar\varepsilon_{ijk}L_k\frac{2}{m_e}H.
\end{aligned} \tag{2}$$

The first two commutators give the quantum mechanical version of statement that $\mathbf{L}$ and $\mathbf{A}$ are conserved (the vanishing commutators imply that the time evolution operator for our system commutes with $\mathbf{L}$ and $\mathbf{A}$). The third commutator is usual angular momentum commutation relation. The fourth commutator follows from the fact that $\mathbf{A}$ is a vector operator (for a more precise discussion of this, see [4]). The last commutation relation is the result of a fairly tedious calculation.

Two more useful facts are the following:

$$\mathbf{A}\cdot\mathbf{L} = \mathbf{L}\cdot\mathbf{A} = 0 \tag{3}$$

$$A^2 \doteq \mathbf{A}\cdot\mathbf{A} = k^2 + \frac{2}{m_e}H\left(L^2 + \hbar^2\right) \tag{4}$$

The first result is trivial from the definition of $\mathbf{A}$. Both of these facts will be vital for us shortly.

Now suppose that there exists a bound state with energy $E < 0$. Let $\mathcal{H}(E)$ be the eigenspace in $\mathcal{H}$ with eigenvalue E. We will restrict the action of all of our operators to this eigenspace[5]. In particular, the restricted Hamiltonian is $H|_{\mathcal{H}(E)} = E$ where E denotes the "multiplication by E" operator (we apologize for overloading notation). In this subspace, we can make sense of the operators $\widetilde{A}_i = \sqrt{\frac{-m}{2E}}A_i$ for $i = x, y, z$. We now define six more operators on $\mathcal{H}(E)$:

$$T_i = \frac{1}{2}(L_i + \widetilde{A}_i) \tag{5}$$

$$S_i = \frac{1}{2}(L_i - \widetilde{A}_i). \tag{6}$$

[5]Because $\mathbf{A}$ and $\mathbf{L}$ commute with the unrestricted Hamiltonian, the action of $\mathbf{A}$ and $\mathbf{L}$ fixes $\mathcal{H}(E)$. That is, if $\psi \in \mathcal{H}(E)$, then $A_i\psi$ and $L_i\psi$ are also in $\mathcal{H}(E)$. This remark allows us to restrict our operators of interest to $\mathcal{H}(E)$.

Note that these operators are Hermitian. The following additional facts about the $\mathbf{T}$ and $\mathbf{S}$ operators will be important for us.

$$
\begin{aligned}
[E, T_i] &= [E, S_i] = 0 && (7)\\
T^2 &= S^2 && (8)\\
[T_i, S_j] &= 0 && (9)\\
[T_i, T_j] &= i\hbar\epsilon_{ijk}T_k && (10)\\
[S_i, S_j] &= i\hbar\varepsilon_{ijk}S_k. && (11)
\end{aligned}
$$

All of these results follow easily from our definitions and from the commutation relations labeled as 2. Note that equation 8 follows from the fact that $\mathbf{A} \cdot \mathbf{L} = \mathbf{L} \cdot \mathbf{A} = 0$.

Equations 9,10, and 11 mean that we have two "angular momentum-like"operators which commute with each other. We can therefore apply raising and lowering operator techniques simultaneous eigenkets of $\mathbf{T}$ and $\mathbf{S}$. It is not the purpose of this paper to discuss this technique in general, so we simply assert a result without proof[6]. The interested reader can consult [4] or, for an extremely deep discussion, [7].

Claim. Suppose that $\psi \in \mathcal{H}(E)$ is a simultaneous eigenvalue of T^2, T_z, and S_z. Then there exists $t \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ such that $T^2\psi = t(t+1)\hbar^2\psi$. Conversely, for each $t \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$, there exists exactly $(2t+1)^2$ independent simultaneous eigenvectors of T^2, T_z, and S_z with eigenvalue $t(t+1)\hbar^2$. Furthermore, these eigenvectors can be denoted by $\{|t, m_t, m_s\rangle : m_t, m_s \in \{-t, -t+1, \ldots, t\}\}$ in such a way that $T_z\,|t, m_t, m_s\rangle = m_t\hbar\,|t, m_t, m_s\rangle$ and $S_z\,|t, m_t, m_s\rangle = m_s\hbar\,|t, m_t, m_s\rangle$.

We now have enough information to find the energy levels of the hydrogen atom. Using equations 3 and 4, the reader can check that

$$\widetilde{A}^2 + L^2 = 4T^2 = \hbar^2 - \frac{k^2 m_e}{2E}.$$

Thus, the action of T^2 on $\mathcal{H}(E)$ is simply multiplication by the constant $\frac{1}{4}\hbar^2 - \frac{k^2 m_e}{8E}$. By our claim above, this eigenvalue must be exactly equal to $t(t+1)\hbar^2$. By solving for E, we arrive at the following result

Fact. *The negative eigenvalues of of the Hamiltonian H (not restricted to $\mathcal{H}(E)$) are in one-to-one correspondence with the eigenvalues $t(t+1)\hbar^2$ of T^2. For a given $t \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\right\}$ the energy eigenvalue is $E_{2t+1} = \frac{-m_e k^2}{2\hbar^2(2t+1)^2}$.*

We can now identify the usual principle quantum number n with $2t+1$. Furthermore, for a given value of of t, our claim above tells us that there are

[6]The proof of the claim may be non-trivial. How do we know, for instance, that there does not exist a third non-trivial operator $\mathbf{R}$ that commutes with $\mathbf{T}$ and $\mathbf{S}$, and satisfies the angular momentum commutation relations? One proof is to solve the hydrogen atom by brute force and reverse engineer the first part of this paper (it will turn out that if a third operator existed, the energy level degeneracy of the hydrogen atom would not be n^2). The claim is undoubtedly related to the fact that in the classical Kepler problem, if we know E, $\mathbf{L}$, and $\mathbf{A}$ for an orbit, we have enough information to fully constrain the orbit.

exactly $(2t+1)^2 = n^2$ linearly independent states in $\mathcal{H}(E_n)$. Thus, our second result is

Fact. *Given a principle quantum number n, the degeneracy of the eigenvalue E_n is n^2.*

The Lie algebra so(4)

If the reader is familiar with the mathematics of Lie groups and Lie algebras, he or she may have noticed that the commutation relations 9,10, and 11 are, up to a constant, the Lie bracket requirements for the Lie algebra $su(2) \oplus su(2)$. More precisely:

Fact. *Let V=Span $\{T_i, S_j | i, j \in \{x, y, z\}\}$ denotes the 6 dimensional real vector space of linear combinations of the operators T_i and S_j (with coefficients in $\mathbf{R}$). Define a function $\{,\} : V \times V \to V$ by $\{P, Q\} = \frac{1}{i\hbar}[P, Q]$. Then, V equipped with this bracket is exactly the Lie algebra $su(2) \oplus su(2)$.*

Now it is easy to show that $su(2) \oplus su(2)$ is isomorphic to the Lie algebra $so(4)$. This is the Lie algebra that generates the Lie group $SO(4)$ of rotations in $\mathbf{R}^4$ (among other Lie groups[7]). We may, therefore, have reason to expect some kind of "hidden symmetry" of the hydrogen atom that will look like rotation in $\mathbf{R}^4$. This turns out to be case, and we will spend most of the rest of this paper discussing this symmetry.

2 The Analytic Approach

The stereographic projection and classical Kepler orbits

In this section, we follow Norcliffe and Percival [2] fairly closely. As with the LRL vector, we will motivate the use of stereographic projection in the hydrogen atom by its use in the classical Kepler problem.

Definition. Let n be a natural number. The *n-sphere* is the set

$$S^n = \left\{\mathbf{x} \in \mathbf{R}^{n+1} | x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

The most familiar examples of spheres are S^1 and S^2 which are the surfaces of the unit circle and the "usual" unit sphere respectively. The sphere which will be of relevance to us is, unsurprisingly, S^3 (which "exhibits $SO(4)$ symmetry").

[7] While a Lie algebra V may generate many Lie groups, all of those Lie groups will "look the same" at the identity. More precisely, suppose G and H are two Lie groups generated by V. Then there exist manifold charts around the identities of each group and a diffeomorphism between those two charts which preserves the group structure.

Definition. View $\mathbf{R}^3$ as the hyperplane $\left\{(x_1, x_2, x_3, 0) \in \mathbf{R}^4\right\}$. Let $\hat{n} = (0,0,0,1) \in \mathbf{R}^4$. The *stereographic projection of* S^3 is the map $\varphi : \mathbf{R}^3 \to S^3$ given by[8]

$$\varphi(x_1, x_2, x_3, 0) = \frac{2\mathbf{x} + (|\mathbf{x}|^2 - 1)\widehat{n}}{|\mathbf{x}|^2 - 1}.$$

We remark now that it is possible to construct a version of classical mechanics in the configuration manifold $\mathbf{R}^4$ rather than the usual three dimensional classical mechanics. Newton's second law is given as usual, except that force has 4 components. The following example will be relevant to us

Example. Consider a free particle constrained to S^3. Its classical trajectories will be "great circles at constant speed," where by a great circle on S^3, we mean the intersection of S^3 with any 3 dimensional hyperplane in $\mathbf{R}^4$ that contains the origin.

Now consider a bound state solution to the classical Kepler problem $\mathbf{r}(t)$. This gives rise to a momentum function $\mathbf{p} : \mathbf{R} \to \mathbf{R}^3$ given by $\mathbf{p}(t) = m\dot{\mathbf{r}}(t)$. Let E be the energy of the Kepler orbit ($E = \frac{\mathbf{p}^2(t)}{2m} - \frac{k}{r(t)}$ for any $t \in \mathbf{R}$). Let $p_E = \sqrt{-2mE}$. Now view the space of momentum states $\mathbf{R}^3$ as a hyperplane embedded in $\mathbf{R}^4$. Let φ denote the stereographic projection defined above, and define a "normalized" stereographic projection $u : \mathbf{R}^3 \to S^3$ as

$$u(\mathbf{p}, 0) = \varphi(\frac{\mathbf{p}}{p_E}, 0) = \frac{2p_E\mathbf{p} + (p^2 - p_E^2)\widehat{n}}{p^2 + p_E^2}.$$

It turns out that the path in S^3 given by $u \circ \mathbf{p}$ (where $\circ$ denotes function composition) is *exactly that of a free particle on* S^3.[9] This fact is shown in detail, via the stationary action principle, by Norcliffe and Percival [2].

The quantum mechanical stereographic projection[10]

We now describe the analogous stereographic projection for the hydrogen atom. Let ψ be a bound state solution to the hydrogen atom. That is, $\psi \in L^2\left(\mathbf{R}^3\right)$

[8] The stereographic projection we have defined has the following geometrical interpretation. Let $(\mathbf{x}, 0)$ be a point in the hyperplane described in this definition. Construct the straight line segment in $\mathbf{R}^4$ that connects $(\mathbf{x}, 0)$ and the "north pole" of S^3. This line segment intersects S^3 at exactly one point besides the north pole. This point is $\varphi(\mathbf{x}, 0)$. We should note that stereographic projections are also sometimes defined in a similar manner except with the sphere placed "on top" of the plane.

[9] This result may be slightly less surprising given the following. If the orbit $\mathbf{r}(t)$ has eccentricity ε, then it can be shown that the momentum vector $\mathbf{p}(t)$ travels through a circle in $\mathbf{R}^3$ with center at a distance $p_E\epsilon\left(1 - \varepsilon^2\right)^{-1/2}$ from the origin and with radius $p_E\left(1 - \varepsilon^2\right)^{-1/2}$ (see [2]). However, it is a fact that the stereographic projection φ takes circles in $\mathbf{R}^3$ to circles on S^3. It would seem however, that we are lucky in that the stereographic projection makes the speed of the path on S^3 constant.

[10] In this section, we will unfortunately omit almost every proof. For this reason, the interested reader is strongly encouraged to look at Bander and Itzykson's paper on the subject [1]. Their paper, which we follow fairly closely, presents this material in greater detail and goes significantly beyond the scope of this paper.

satisfying $H\psi = E\psi$ for some $E < 0$, where $H = \frac{p^2}{2m} - \frac{k}{r}$. As before, let $p_E = \sqrt{-2mE}$. Motivated, once again, by the classical Kepler problem, we consider the momentum-space wave function. This is an element $\Phi \in L^2\left(\mathbf{R}^3\right)$ given by[11]

$$\Phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbf{R}^3} \psi(\mathbf{x}) e^{-\mathbf{i}\mathbf{p}\cdot\mathbf{x}/\hbar} d^3\mathbf{x}. \tag{12}$$

Now we wish to let this momentum-space wave function undergo a stereographic projection analogous to the function u defined in the last section. In order to make sense of this problem, we first remark that S^3, being a smooth 3-manifold embedded in $\mathbf{R}^4$, has an induced three dimensional Lebesgue measure (the same is true for S^2, and in this case that measure simply tells us how to measure the areas of regions on S^2). Given this measure, we can now perform Lebesgue integration on S^3 and can thus consider the Hilbert space $L^2(S^3)$.

Let u denote the stereographic transformation $\mathbf{R}^3 \to S^3$ discussed in the last section. We now define a map $\xi : L^2(\mathbf{R}^3) \to L^2(S^3)$ which will "preserve probability" with respect to the transformation u. Because u is one-to-one, we may define

$$(\xi\Phi)(u(\mathbf{p})) = \frac{1}{p_E^{5/2}} \left(\frac{p_E^2 + p^2}{2}\right)^2 \Phi(\mathbf{p}).$$

Now the "coordinate transformation" u induces the transformation of measure (we omit the proof):

$$d\Omega = \frac{(2p_E)^3}{(p_E^2 + p^2)^3} d^3\mathbf{p}.$$

Using this fact, the following can be shown (this is the reason that we consider the definition of ξ to be "correct")

Claim. Let $\Phi \in L^2(\mathbf{R}^3)$ and let ξ be the map defined above. If V is a measurable subset of S^3 which does not intersect the north pole (0,0,0,1), then

$$\int_V |(\xi\Phi)|^2 d\Omega = \int_{u^{-1}(V)} \Phi d^3\mathbf{p}.$$

Now in the classical Kepler problem, the stereographic projection of $\mathbf{p}(t)$ behaved like a free particle on S^3. The exactly analogous thing happens for the stereographic projection of the momentum-space wavefunction. The reader may not be familiar with the quantum mechanical free particle on S^3, so we briefly describe it.

Definition. Fix a natural number n. Let $\mathcal{F}$ denote the smooth functions from $\mathbf{R}^n$ to $\mathbf{C}$. The Laplacian in $\mathbf{R}^n$ is the map $\nabla^2_{\mathbf{R}^n} : \mathcal{F} \to \mathcal{F}$ defined as $\nabla^2_{\mathbf{R}^n} f = \frac{\partial^2 f}{\partial x_n^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}$.

[11] The reader need not worry about the convergence properties of the Fourier transform of ψ. It is known that hydrogen wave functions fall off exponentially with r. This means that ψ is in the *Schwartz space of* $\mathbf{R}^3$ which is, very roughly, a space of functions $\mathbf{R}^3 \to \mathbf{C}$ which become small sufficiently quickly to behave very well under Fourier transformation.

Fact. *Let $n > 1$. The Laplacian in $\mathbf{R}^n$ can be decomposed as*

$$\nabla^2_{\mathbf{R}^n} = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla^2_{S^{n-1}}$$

where $\nabla^2_{S^{n-1}}$ is an operator that can be written only in terms of angular coordinates and partial derivatives with respect to angular coordinates. The operator $\nabla^2_{S^{n-1}}$ can therefore be viewed as an operator on smooth functions defined on S^{n-1}.

We call the operator above the Laplace-Beltrami operator on S^{n-1}. The reader can view this operator as "the angular part of the Laplacian on $\mathbf{R}^n$.

Example. The Laplacian on $\mathbf{R}^3$ is (in usual spherical coordinates), $\nabla^2_{\mathbf{R}^3} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)$. The Laplace-Beltrami operator on S^2 is $\nabla^2_{S^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$. If $\mathbf{L}$ is the orbital angular momentum operator in $\mathbf{R}^3$, it is a fact that $L^2 = -\nabla^2_{S^2}$.

Now in complete analogy to the free particle on S^2, we view the "free particle on S^3" as a solution $f \in L^2(S^3)$ satisfying the Schrödinger equation

$$\nabla^2_{S^3} f = \alpha f$$

for some energy eigenvalue α. Fortunately, differential equations of this form are very well studied:

Fact. *Fix a natural number n. If $f \in L^2(S^n)$ satisfies $\nabla^2_{S^n} f = \alpha f$* for some $\alpha \in \mathbf{C}$, then

1) $\alpha = -\lambda(\lambda + n - 1)$ for some $\lambda \in \{0, 1, 2, 3, \ldots\}$,

2)there are exactly $\begin{pmatrix} \lambda + n \\ \lambda \end{pmatrix} - \begin{pmatrix} \lambda + n - 2 \\ \lambda - 2 \end{pmatrix}$ linearly independent eigenvectors with the same eigenspace, and

3)f is called a spherical harmonic (on S^n) and is denoted $Y^{(n)}_{\lambda,q}$ where λ is as above, and q is another integer to label a basis for the eigenspace of λ.

Example. The usual spherical Harmonics on S^2 are, in the above notation, denoted $Y^{(2)}_{l,m}$ where $l \in \{0, 1, 2, \ldots\}$ and m takes on $2l + 1$ values. We make the correspondence $l \leftrightarrow \lambda$ and $m \leftrightarrow \alpha$. These spherical harmonics satisfy $L^2 Y^{(2)}_{l,m} = l(l+1)\hbar^2 Y^{(2)}_{l,m}$. We remarked in the last example that $L^2 = -\nabla^2_{S^2}$. Thus, this new definition of spherical harmonics is compatible with our old definition. Furthermore, part 2 of the fact above confirms that m can take on $2l + 1$ values because

$$\begin{pmatrix} l + 2 \\ l \end{pmatrix} - \begin{pmatrix} l + 2 - 2 \\ l - 2 \end{pmatrix} = 2l + 1.$$

We now return to our discussion of the hydrogen atom. It can be shown that under the transformation $\Phi \to \xi\Phi$ of the momentum space wavefunction for the hydrogen atom, the transformed function will satisfy $\nabla^2_{S^3}(\xi\Phi) = \alpha(\xi\Phi)$

so $\xi\Phi = Y_{\lambda,q}^{(n)}$. Thus, exactly as in the classical Kepler problem, the electron's momentum behaves like a free particle on S^3. We can obtain the hydrogen wave functions by performing ξ^{-1} followed by the inverse Fourier transform to undo equation 12. We remark finally that the integer λ corresponds to the principle quantum number n in the hydrogen atom (see [1] for a proof). The second part of the fact above gives the correct degeneracy of the eigenvalue E_n because

$$\begin{pmatrix} \lambda+3 \\ \lambda \end{pmatrix} - \begin{pmatrix} \lambda+3-2 \\ \lambda-2 \end{pmatrix} = (\lambda+1)^2 = n^2.$$

Concluding remarks

We have described two very distinct methods for analyzing the hydrogen atom, both of which have some relation to $SO(4)$ symmetry (the second example describes that symmetry very explicitly). There is, in fact, a sense in which the Lie algebra obtained in the first part of this paper generates the symmetry in the second part. This is discussed in detail by Bander and Itzykson [1] (see their section on parabolic coordinates). It is pleasing that both of our approaches have classical analogues. Furthermore, each approach gives a new perspective on the "accidental degeneracy" of the hydrogen atom's energy levels.

We finally remark that this symmetry is not limited to the problem discussed here. It has recently been shown by Chen, Deng, and Hu [8] that the relativistic hydrogen atom also posseses an SO(4) symmetry.

References

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