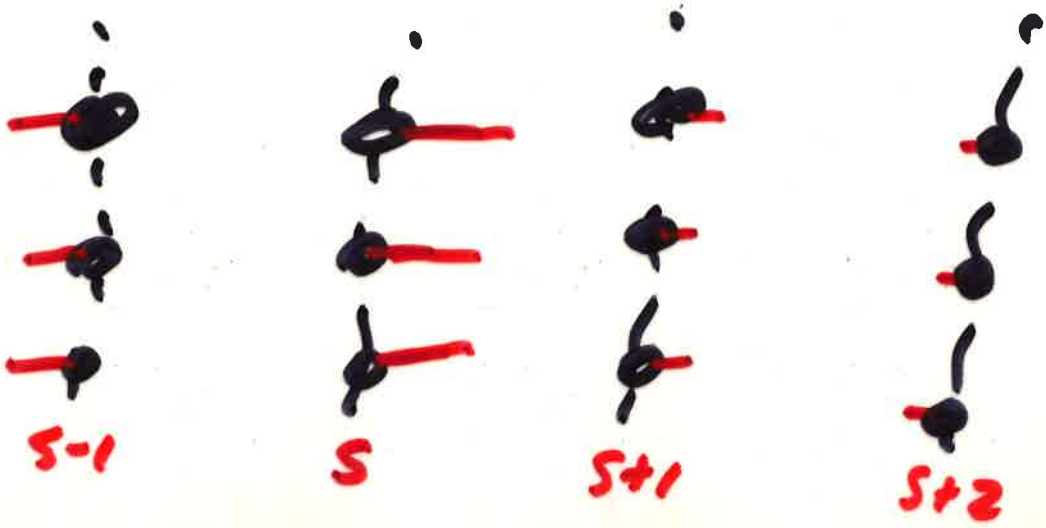


$\lambda = \frac{a\sqrt{3}}{3}$

For k's outside the Brillouin zone $|k| > \pi/a$

$$u_s = Ae^{ikas} = Ae^{ias(k_{in} + 2\pi n)}$$

$$= Ae^{ias(k_{in})}$$

Long wavelength limit (small k limit)

D.R. $\omega^2 = \frac{2c}{m}[1 - \cos(ka)]$

$\lambda \gg a$, $ka \ll 1$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots$$

$$\omega^2 = \frac{2C}{m}\left[1 - \left(1 - \frac{k^2a^2}{2}\right)\right] + \dots$$

$$\omega^2 = \frac{2C}{m}\frac{k^2a^2}{2}$$

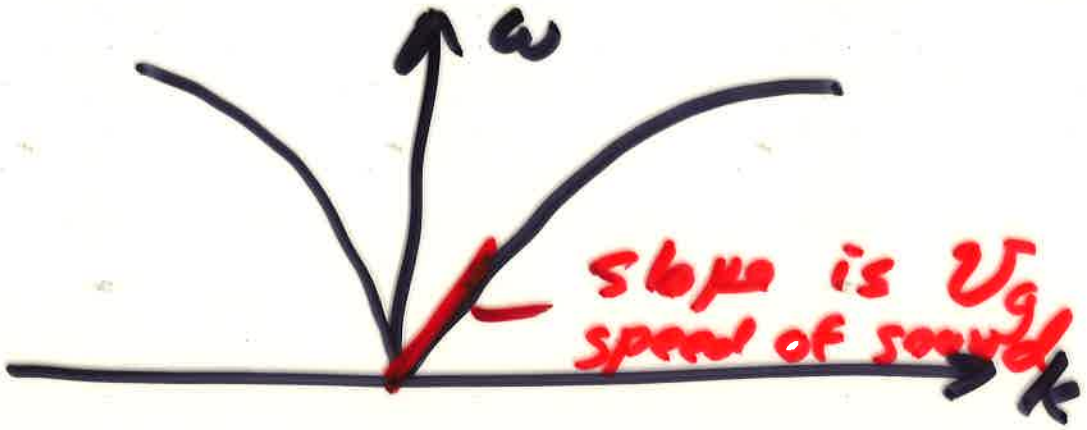
$$\omega = a\sqrt{\frac{C}{m}}\,k \qquad \omega \propto k$$

no dispersion

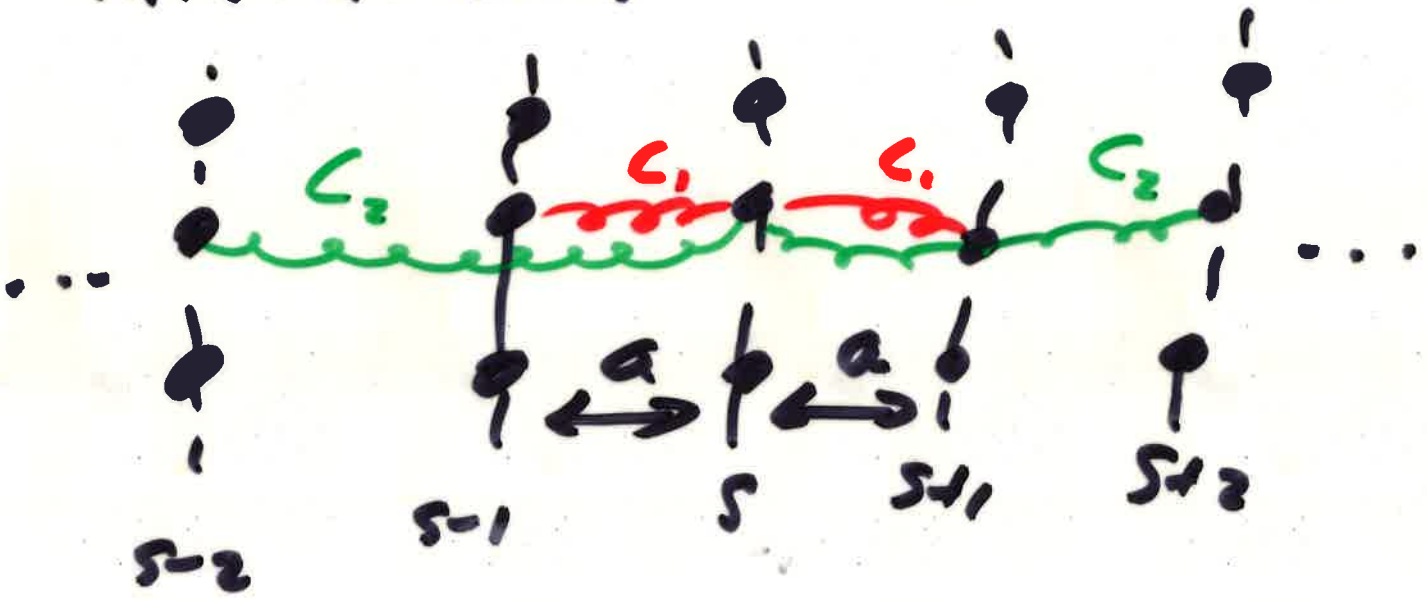
Speed of sound (long λ's)

no dispersion : $v_g = v_\phi$

$$v_g = \frac{\omega}{k} = a\sqrt{\frac{C}{m}} = v_\phi = \frac{d\omega}{dk}$$



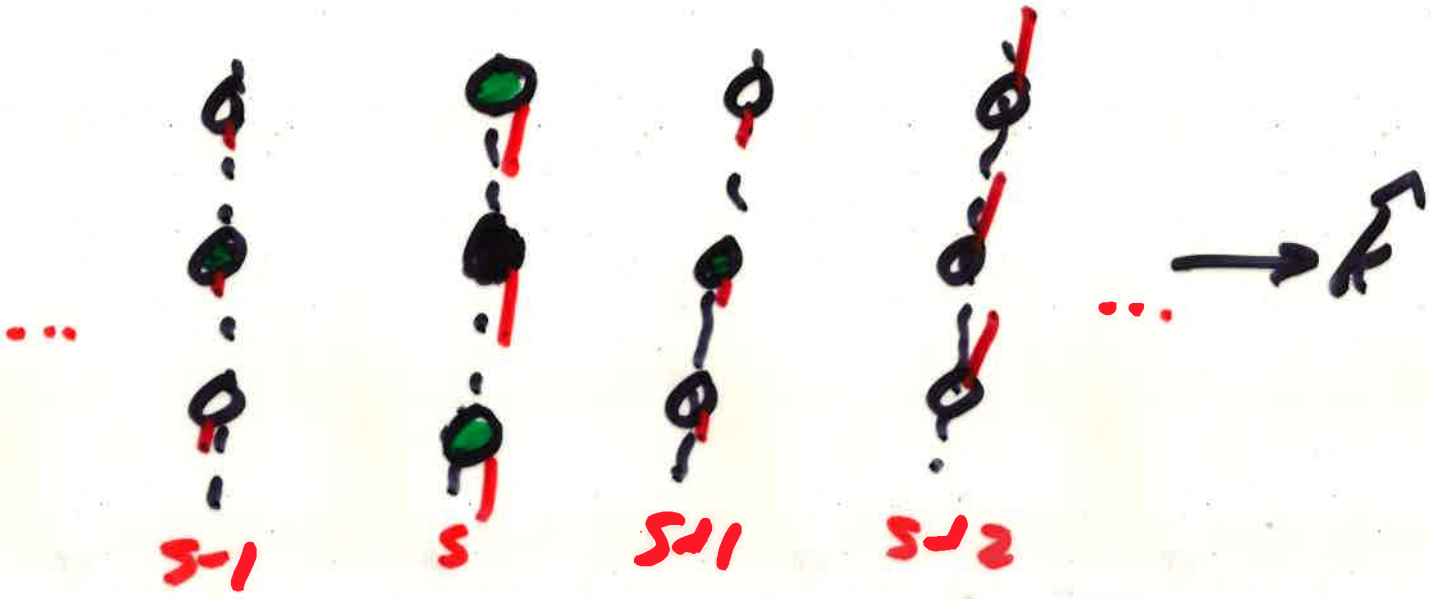
Longitudinal waves
more than nearest neighbor
interactions



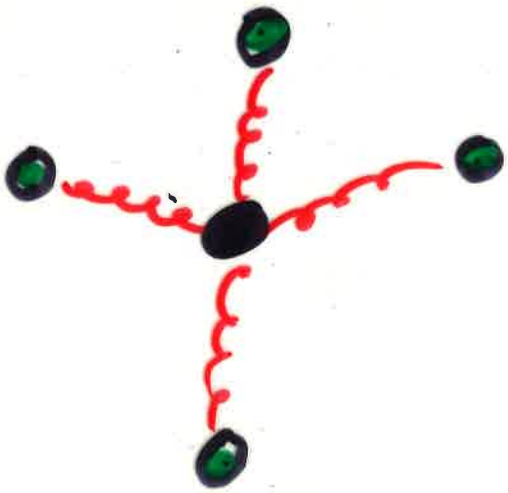
$$\omega^2 = \frac{2C}{m}\left[1 - \cos(ka)\right]$$

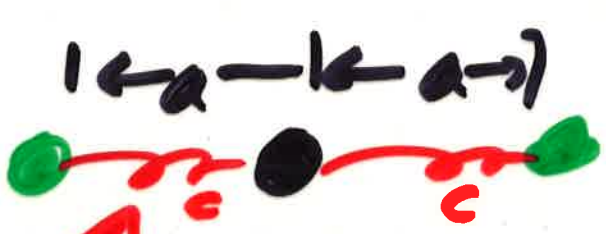
$$\omega^2 = \frac{2}{m}\sum_{p=1}^{2} C_p\left[1 - \cos(kpa)\right]$$

Transverse waves



No contribution from nearest neighbor interactions





If these springs are under tension (rest length $<a$) there is a restoring force proportional to displacement.

$$F \propto C_{eff}\, u$$

Our problem

Rest length of springs = a

$$F \propto u^3$$

new length is $\sqrt{a^2+\Delta u^2}$

amount spring stretched is

$$\sqrt{a^2+\Delta u^2} - a \qquad \Delta u \ll a$$

$$= a\sqrt{1+\frac{\Delta u^2}{a^2}} - a$$

$$= a\left(1+\frac{\Delta u^2}{a^2}\right)^{1/2} - a$$

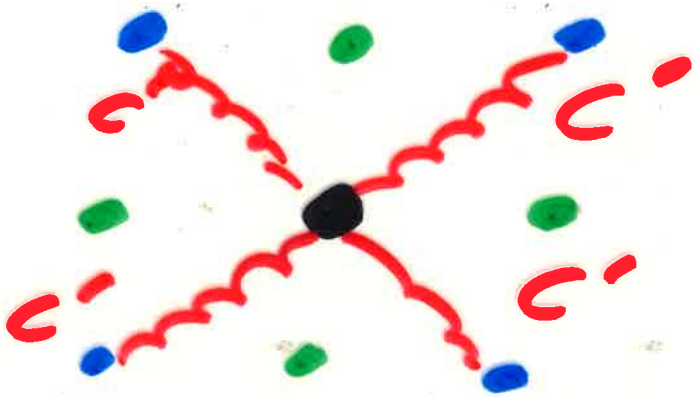
$$= a\left[1+\frac{\Delta u^2}{2a^2}+\cdots\right] - a$$

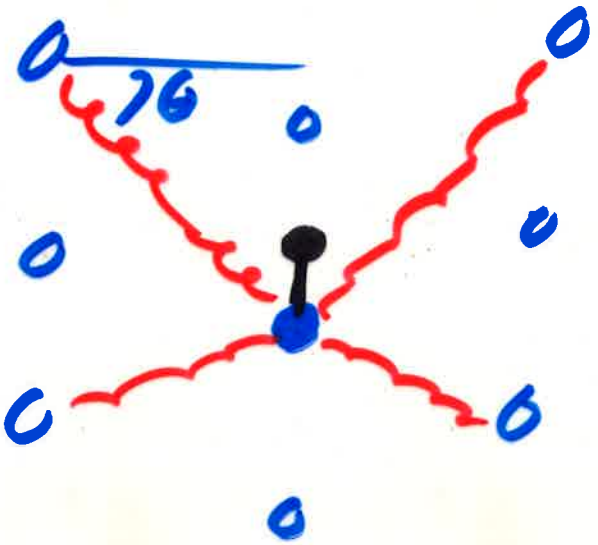
$$= \frac{\Delta u^2}{a}$$

$$\text{Force} = F = C\frac{\Delta u^2}{a}$$

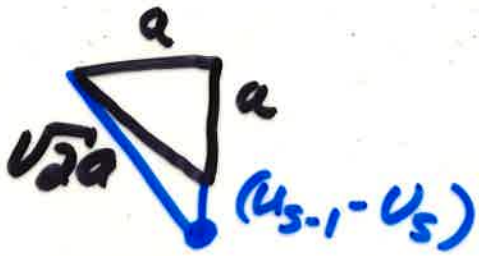
$$F_y = F\sin\theta \approx F\tan\theta \simeq F\frac{\Delta u}{a}$$

$$2F_y = F_{net} = C\frac{\Delta u^3}{a^2} \propto \Delta u^3$$





Left side



new length of spring is

$$\sqrt{a^2 + [a + (u_{s-1} - u_s)]^2}$$

$$= \sqrt{2a^2 + 2a^2\frac{(u_{s-1} - u_s)}{a} + (u_{s-1} - u_s)^2}$$

$$= \sqrt{2}\,a\left[1 + \frac{(u_{s-1} - u_s)}{a}\right]^{1/2}$$ binomial theorem

$$(1+x)^n \approx 1 + nx + \dots - O(x^2)$$

$$= \sqrt{2}\,a\left[1 + \frac{(u_{s-1} - u_s)}{2a} + \dots\right]$$

spring has been stretched $\dfrac{(u_{s-1} - u_s)}{\sqrt{2}}$

force $F = C'\frac{(u_{s-1} - u_s)}{\sqrt{2}}$ $\sin 45° = \frac{1}{\sqrt{2}}$

vertical component $F_y = F\sin\theta = C'\frac{(u_{s-1} - u_s)}{2}$

Now both left springs

$$F_y = C'(u_{s-1} - u_s)$$

Both right springs

$$F_y = C'(u_{s+1} - u_s)$$

Total force 4 springs $= C'[u_{s+1} + u_{s-1} - 2u_s]$

same form as longitudinal wave

Long. - nearest neighbors

Trans. - next to nearest neighbors