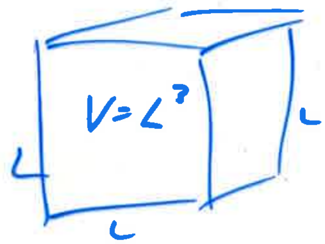
Bose-Einstein Condensation BEC

Spin-0 bosons



$$\lambda_x = \frac{2L}{n_x} \qquad p_x = \frac{h}{\lambda_x}$$

$$E = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

Particle in a box

Ground state energy $\epsilon_0 = \frac{h^2}{8mL^2}(1^2+1^2+1^2) = \frac{3h^2}{8mL^2}$

At temp. T, the average number of atoms in the ground state is?

$$N_0 = \frac{1}{e^{+\beta(\epsilon_0-\mu)} - 1}$$

At low T, N_0 will be large.

$$\Rightarrow e^{+\beta(\epsilon_0-\mu)} \simeq 1 \Rightarrow \beta(\epsilon_0-\mu) \simeq 0$$

Taylor series

$$N_0 = \frac{1}{[1 + \beta(\epsilon_0-\mu) + \cdots] - 1} = \frac{k_B T}{\epsilon_0 - \mu}$$

At $T=0$, μ must be $\approx \epsilon_0$: $N_0 = \frac{0}{0}$

$N_0 > 0 \qquad \mu \lesssim \epsilon_0$

At $T \neq 0$ but small, still have large N_0

$$\mu = \epsilon_0 - \delta$$

$$N_0 = \frac{k_B T}{\epsilon_0 - (\epsilon_0 - \delta)} = \frac{k_B T}{\delta} = \frac{\text{small}}{\text{very small}} = \text{large}$$

$$\delta \ll k_B T$$

Total Number of Particles

$$N = \sum_i N_i = \sum_i \frac{1}{e^{+\beta(\epsilon_i - \mu)} - 1}$$

microstates

$$\rightarrow \int_{\epsilon=0}^{\infty} \frac{g(\epsilon)\, 1}{e^{+\beta(\epsilon - \mu)} - 1} d\epsilon$$

density of states = degeneracy
many microstates have the same energy.

$g(\epsilon)$ same as fermions, but one spin state instead of 2

$$g(\epsilon) = \frac{\pi}{4} \frac{(8m)^{3/2}}{h^3} V\sqrt{\epsilon} = \frac{2}{\sqrt{\pi}} \left(\frac{2\pi m}{h^2}\right)^{3/2} V\sqrt{\epsilon}$$

We know μ should be $\epsilon_0 - \delta$, $\rightarrow$ Guess

$$\mu \approx 0$$

$$N \simeq \int_{\epsilon=0}^{\infty} \frac{g(\epsilon)}{e^{+\beta\epsilon}-1}\, d\epsilon$$

change variable
$x = \beta\epsilon$
$dx = \beta\, d\epsilon$

$$= \frac{2}{\sqrt{\pi}}\left(\frac{2\pi m}{h^2}\right)^{3/2} V \int_{\epsilon=0}^{\infty} \frac{\sqrt{\epsilon}\, d\epsilon}{e^{\beta\epsilon}-1}$$

Reimann zeta function

$\simeq 2.612$

$$= \frac{2}{\sqrt{\pi}}\underbrace{\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}}_{\frac{1}{\lambda_Q^3}} V \int_{x=0}^{\infty} \frac{\sqrt{x}\, dx}{e^{x}-1} = \frac{\sqrt{\pi}}{2}\zeta\left(\tfrac{3}{2}\right)$$

$$N \simeq \zeta\left(\tfrac{3}{2}\right)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} V = \zeta\left(\tfrac{3}{2}\right)\frac{V}{v_Q}$$

independent of T — depends on T

There is a $T = T_c$ for which this is true

critical, condensation

$$N = \zeta\left(\tfrac{3}{2}\right)\left(\frac{2\pi m k_B T_c}{h^2}\right)^{3/2} V$$

$n = \frac{N}{V}$ — number density

$$T_c = \frac{h^2}{2\pi m k_B}(n)^{2/3}\left(\zeta\left(\tfrac{3}{2}\right)\right)^{-2/3}$$

$\leftarrow 0.527$

$$N^* = N - N_0 = \zeta\left(\tfrac{3}{2}\right)\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} V \qquad \text{for } T < T_c$$

$\uparrow$ # of atoms in excited states

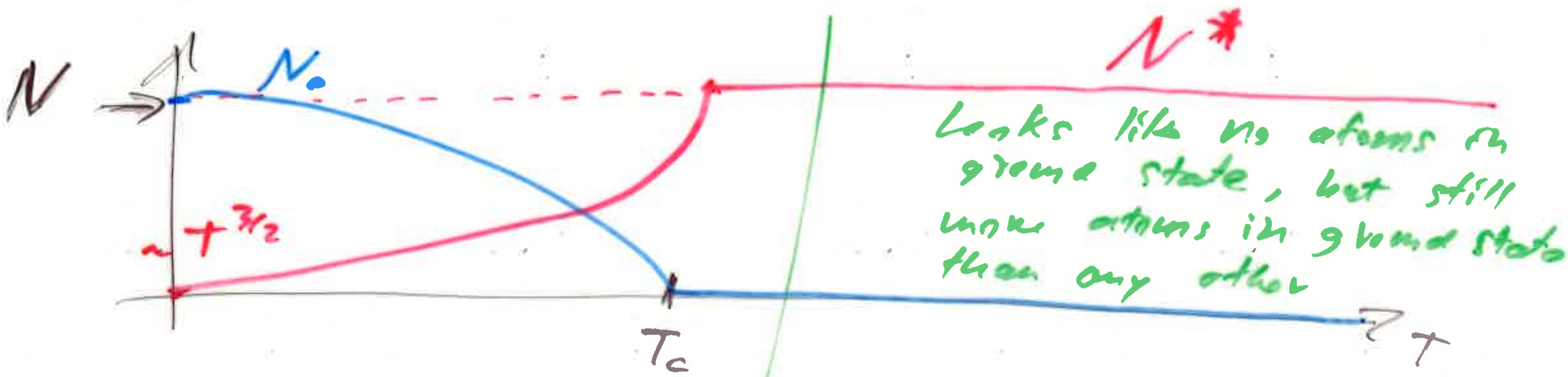
For $T > T_c$, $\mu < 0$ and almost all of the atoms are in excited states

$$\frac{N_0}{N} \ll 1$$

For $T < T_c$, $\mu \approx 0$ and $N^* = \left(\frac{T}{T_c}\right)^{3/2} N$

$$N_0 = N - N^* = N\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right]$$

N_0 ↑ # in ground state; N ↑ total # atoms; N^* ↑ # in all excited states



$$N_0 > N_1 > N_2 > N_3 \ldots$$

$$N_0 \ll N_1 + N_2 + N_3 + \ldots$$