1. Starting with the definition of the factorial

$$N! = \int_{x=0}^{\infty} x^N \, e^{-x} \, dx = \Gamma(N+1)$$

show all the steps to derive the formula

$$\int_{r=0}^{\infty} r^{N-1} \, \exp(-r^2) \, dr = \frac{1}{2}\Gamma\left(\frac{N}{2}\right)$$

that we used in the lecture notes.

2. Remember our convention for n-spheres: $n = 2$ dimensions corresponds to the film of a soap bubble and has surface area $S_2 = 4\pi r^2$; the volume of the solid spherical ball is $V_3 = \frac{4}{3}\pi r^3$; the "volume" of a disk is $V_2 = \pi r^2$. (n is always the power of r).

   (a) For which (non-integer) dimension n is the area of the unit ($r = 1$) hypersphere a maximum?

   (b) For which (non-integer) dimension n is the volume of the unit hypersphere a maximum?

   (c) Plot graphs of S_n and V_n versus n.

3. (a) Using last week's problem on the entropy of a black hole, find the temperature of a black hole as a function of its mass.

   (b) Find the temperature of a solar-mass black hole.

   (c) Find the temperature of the largest known black hole.

   (d) Find the temperature of a Planck-mass black hole.

4. Consider a two-state paramagnet with $N = 10^{23}$ elementary dipoles, with the total energy fixed at zero so that exactly half of the dipoles point up and half point down. Give all answers first as a function of N, then numerically.

   (a) How many total states (with any energy) does the system have?

   (b) How many microstates have zero energy? Give the exact formula in terms of N, then use Stirling's approximation.

   (c) What fraction of states have energy zero? (100% is the wrong answer.)

   (d) Suppose that the microstate can change a googol (10^{100}) times every second. How long will it take the system to explore all the zero-energy microstates. Compare this to the age of the Universe, 14 billion years.

   (e) Comment on the statement, "If you wait long enough, the system eventually will be found in any of its accessible microstates."

Bonus

1. According to the Sackur-Tetrode equation (which we will derive soon),

$$S = Nk_B \left[\ln \left(\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right]$$

the entropy of a monatomic ideal gas can become negative when its temperature (and hence its energy) is sufficiently low. Of course this is absurd, so the Sackur-Tetrode equation must be invalid at very low temperatures. Suppose that you start with a sample of helium at room temperature and atmospheric pressure, then lower the temperature while holding the density fixed. Pretend that the helium remains a gas and does not liquefy. Below what temperature would the Sackur-Tetrode equation predict that S is negative?

2. (a) Consider a horizontal slab of atmosphere whose thickness is dz. Find an expression for $\frac{dP}{dz}$ in terms of the average mass m of an air molecule and the number density $n = N/V$.
   (b) Use the ideal gas law to find a differential equation for the pressure.
   (c) Assume that the temperature does not vary with height and solve the differential equation for the pressure as a function of height.
   (d) Explain the boundary condition at z=0.