1. Read Kardar chapter 7. Did you read all the pages?

2. Write the formula for the partition function Z of an electron in a hydrogen atom. If the sum is finite, to what value does it converge? If the sum is infinite, prove it.

3. (a) Estimate the moment of inertia I for a CO molecule rotating about an axis through its center of mass and perpendicular to the line joining the atoms. Are the electrons or the nuclei more important?
   (b) At what temperature T would you expect this rotational degree of freedom to freeze out?
   (c) Estimate the moment of inertia I for a CO molecule rotating about an axis through its center of mass and parallel to the line joining the atoms. Are the electrons or the nuclei more important?
   (d) At what temperature T would you expect this rotational degree of freedom to freeze out?
   (e) Explain why we ignore the latter degree of freedom.

4. Consider a diatomic molecule with non-identical atoms like CO. Define the energy $\epsilon \equiv \frac{\hbar^2}{2I}$. Use a computer to sum the rotational quantum mechanical partition function numerically, keeping terms through $\ell = 10$. Calculate the average energy, and then get the heat capacity per diatomic molecule in units of k_B. Plot (not sketch) the heat capacity versus x, where $x \equiv k_B T/\epsilon$, for $x = 0$ to 7. Do you see the bump that I drew in lecture?

5. The pressure of an ideal gas can be expanded as a power series in n/n_Q, where $n = N/V$ is the number density or concentration and $n_Q = (mk_BT/2\pi\hbar^2)^{3/2}$ is the quantum concentration (inverse thermal de Broglie wavelength cubed). The first two terms in this expansion have the form

$$P = nk_BT\left(1 + \alpha\frac{n}{n_Q} + \cdots\right).$$

   In this problem, you will compute the number α for both fermions and bosons.

   (a) The Fermi-Dirac or Bose-Einstein distribution function $< n >$ can be expanded as a power series in $x \equiv \exp[-\beta(\epsilon - \mu)]$. Find the first two terms in this expansion for both fermions and bosons. (Hint: do the two cases at the same time using the $\pm$ sign.)

(b) Using the approximation above, evaluate the particle number

$$N = \int_0^\infty d\epsilon g(\epsilon) < n >,$$

where $g(\epsilon) = \frac{V\sqrt{\epsilon}}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$ is the density of states. The Gaussian integrals that arise can be looked up to save time. What is n/n_Q to this order for both fermions and bosons?

(c) Calculate the chemical potential μ for both fermions and bosons. Expand in the small quantity n/n_Q.

(d) Recalling that $\mu = (\partial F/\partial N)_{T,V}$, integrate to find the Helmholtz free energy $F(T, V, N)$ for both fermions and bosons.

(e) Now calculate the pressure by taking the appropriate partial derivative of F and find α for both fermions and bosons. In one case the first quantum correction makes the pressure smaller than for a classical ideal gas with the same concentration and temperature, and in the other case the first quantum correction makes the pressure larger. Which is which?

Bonus

1. Solve the attached problem on white dwarfs.

Problem 7.23. A **white dwarf** star (see Figure 7.12) is essentially a degenerate electron gas, with a bunch of nuclei mixed in to balance the charge and to provide the gravitational attraction that holds the star together. In this problem you will derive a relation between the mass and the radius of a white dwarf star, modeling the star as a uniform-density sphere. White dwarf stars tend to be extremely hot by our standards; nevertheless, it is an excellent approximation in this problem to set $T = 0$.

(a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass M, radius R) must equal

$$U_{\text{grav}} = -(\text{constant})\frac{GM^2}{R},$$

where (constant) is some numerical constant. Be sure to explain the minus sign. The constant turns out to equal 3/5; you can derive it by calculating the (negative) work needed to assemble the sphere, shell by shell, from the inside out.

(b) Assuming that the star contains one proton and one neutron for each electron, and that the electrons are nonrelativistic, show that the total (kinetic) energy of the degenerate electrons equals

$$U_{\text{kinetic}} = (0.0088)\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}.$$

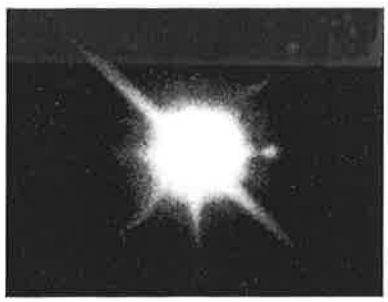

Figure 7.12. The double star system Sirius A and B. Sirius A (greatly overexposed in the photo) is the brightest star in our night sky. Its companion, Sirius B, is hotter but very faint, indicating that it must be extremely small—a white dwarf. From the orbital motion of the pair we know that Sirius B has about the same mass as our sun. (UCO/Lick Observatory photo.)

The numerical factor can be expressed exactly in terms of π and cube roots and such, but it's not worth it.

(c) The equilibrium radius of the white dwarf is that which minimizes the total energy $U_{\text{grav}} + U_{\text{kinetic}}$. Sketch the total energy as a function of R, and find a formula for the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?

(d) Evaluate the equilibrium radius for $M = 2 \times 10^{30}$ kg, the mass of the sun. Also evaluate the density. How does the density compare to that of water?

(e) Calculate the Fermi energy and the Fermi temperature, for the case considered in part (d). Discuss whether the approximation $T = 0$ is valid.

(f) Suppose instead that the electrons in the white dwarf star are highly relativistic. Using the result of the previous problem, show that the total kinetic energy of the electrons is now proportional to $1/R$ instead of $1/R^2$. Argue that there is no stable equilibrium radius for such a star.

(g) The transition from the nonrelativistic regime to the ultrarelativistic regime occurs approximately where the average kinetic energy of an electron is equal to its rest energy, mc^2. Is the nonrelativistic approximation valid for a one-solar-mass white dwarf? Above what mass would you expect a white dwarf to become relativistic and hence unstable?

Problem 7.24. A star that is too heavy to stabilize as a white dwarf can collapse further to form a **neutron star**: a star made entirely of neutrons, supported against gravitational collapse by degenerate neutron pressure. Repeat the steps of the previous problem for a neutron star, to determine the following: the mass-radius relation; the radius, density, Fermi energy, and Fermi temperature of a one-solar-mass neutron star; and the critical mass above which a neutron star becomes relativistic and hence unstable to further collapse.