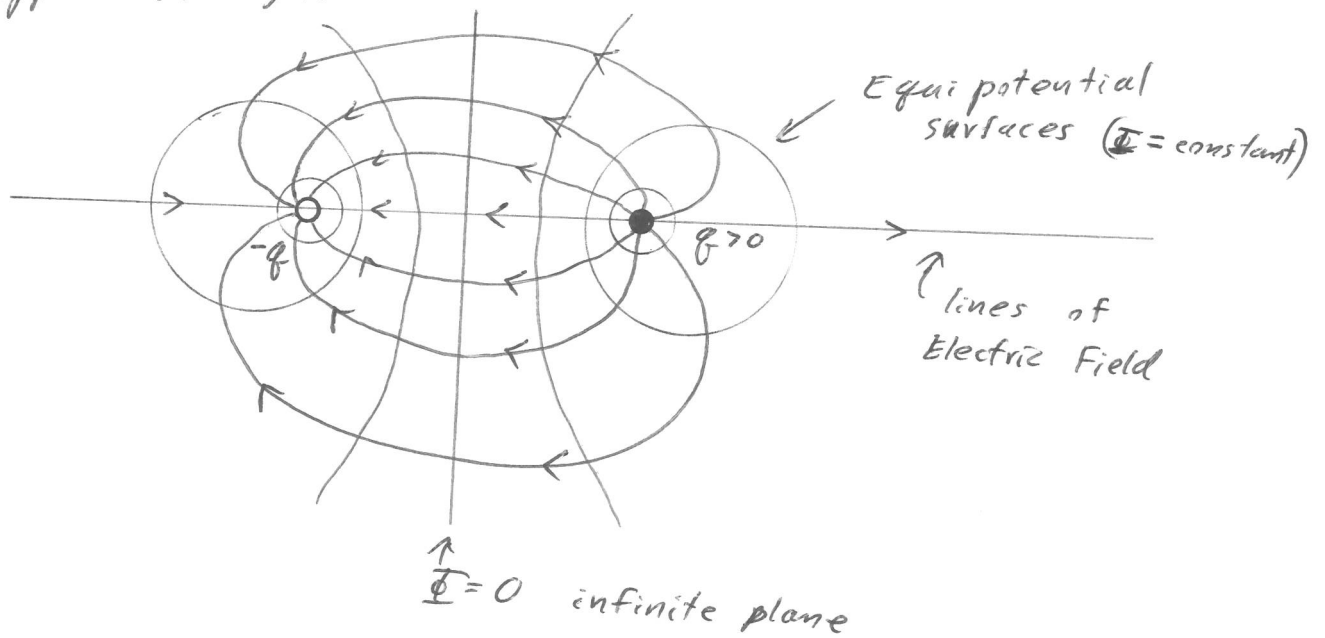


19 Sept 95

D) The Method of Images

Consider two point charges equal in magnitude, but opposite in sign.



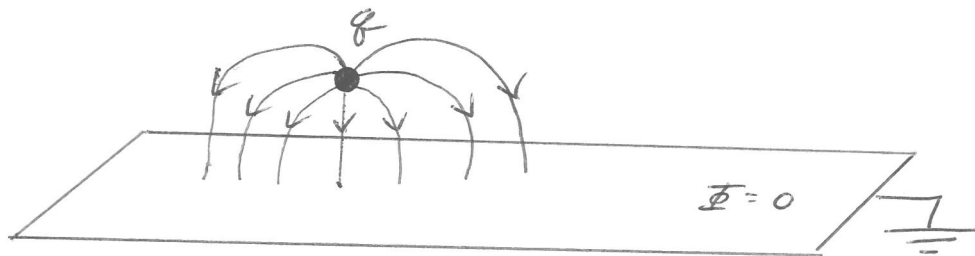
The 3-dimensional equipotential surfaces are quartic in x and y (not quadratic like ellipsoids)

The symmetry plane is an equipotential surface on which $\Phi = 0$.

By the way, the lines of electric field penetrate the equipotential surfaces at right angles.

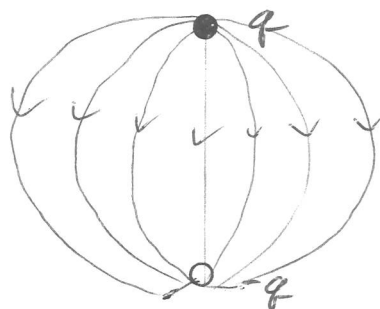
After this brief review, we are ready to look at two distinct problems!

First, consider the problem of finding the potential in the upper half space (UHS) for a point charge sitting a distance z' above an infinite grounded conducting plane.



The point charge q induces a charge density on the conducting plane. Lines of electric field emanate radially from the point charge and intersect the plane of the conductor at right angles.

Compare this situation with an arrangement of two point charges $+q$ and $-q$ with no conducting plane:



It is clear that in the UHS, the two problems are identical, that is the potentials for $z > 0$ are exactly the same (and hence so are the electric fields).

It is just as clear that in the Lower HS, the two problems are radically different.

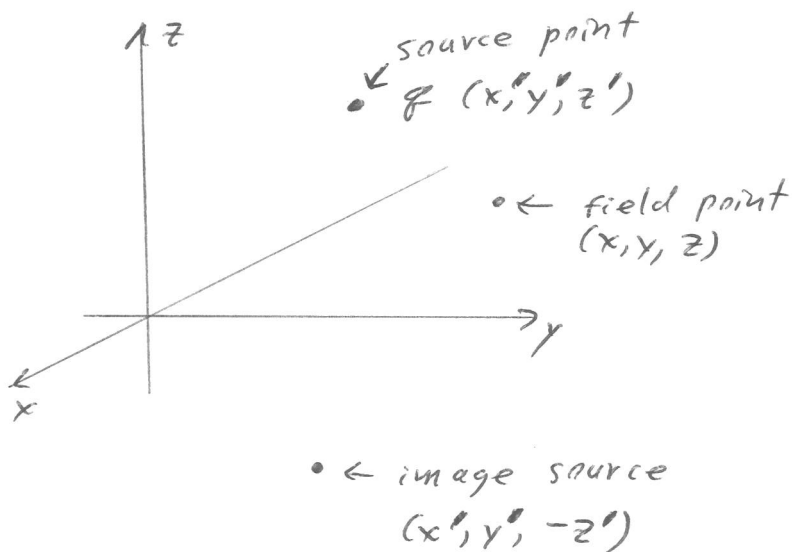
For example, $\Phi = 0$ for $z < 0$ in the conducting plane problem, And $\Phi \neq 0$ for $z < 0$ in the two charge problem.

Usually, we are interested in one Half Space at a time, not both. Suppose the Volume V in which we would like to determine the potential is the UHS.

The bounding surface S is composed of the infinite conducting plane and the surface at infinity. We know nothing about the charge distribution outside V . To obtain an answer for the potential in the UHS, we can postulate charges outside V ; these are called image charges.

Let's solve the point charge and conducting plane problem with an image charge in the LHS.

Replace the conducting plane with charge $-q$ at $-z'$.



$$\Phi(\vec{r}) = \frac{q}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} + \frac{-q}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

this potential is valid for $z > 0$ only!

It satisfies Poisson's Equation in the RHS

$$\nabla^2 \Phi(\vec{r}) = -4\pi q \delta^3(\vec{r} - \vec{r}') \quad \text{for } z > 0 \text{ only}$$

and is satisfied the Dirichlet boundary conditions:

$$\Phi(x, y, 0) = 0$$

$$\Phi(\vec{r}) \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

Since $\Phi(\vec{r})$ is a solution, it is the solution since the solution is unique.

Now that we know the potential in the UHS we have essentially the Dirichlet Green function for the UHS geometry.

Remember $G_0(\vec{r}, \vec{r}')$ is the potential at \vec{r} due to a unit point charge at \vec{r}' and the value of $G_0(\vec{r}, \vec{r}')$ on the boundary S must be zero.

(By symmetry $G_0(\vec{r}, \vec{r}')$ is also the potential at \vec{r}' due to a unit point charge at \vec{r} .)

$$G_{\text{UHS}}(\vec{r}, \vec{r}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}$$

This is the potential Φ with $q=1$. [Note $G_0(\vec{r}, \vec{r}') = G_0(\vec{r}', \vec{r})$]

Now that we have the Green function, we can solve any problem with Dirichlet boundary conditions in the UHS.

$$\Phi(\vec{r}) = \int_V dV' \rho(\vec{r}') G_0(\vec{r}, \vec{r}') - \oint_S \frac{dS'}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}_{\vec{r}'} G_0(\vec{r}, \vec{r}')$$

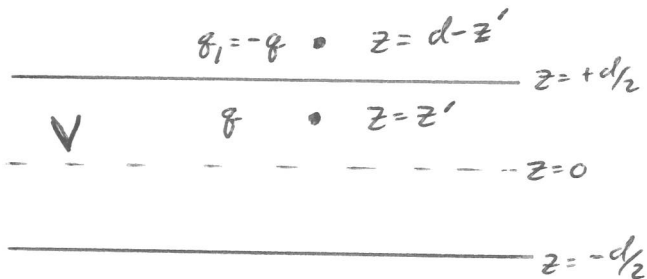
where $\rho(\vec{r}')$ is in the UHS and $\Phi(\vec{r}')$ on S is completely arbitrary.

Let's try the more complicated example of two parallel grounded infinite conducting planes with a point charge between them. We will choose the origin such that the plates are at $z = \pm d/2$ and the point charge is at z' . The volume V is the space between the plates and the bounding surface S is composed of the two planes. Any image charges must be placed outside V , that is above $z = d/2$ or below $z = -d/2$.

The general rule in placing an image charge is that a charge and its image are equidistant above and below a conducting plane and the image will have opposite sign. This will guarantee that the conducting planes are maintained at zero potential. Remember that the planes are removed when the image charges are in place.

$$q_3 = -q \cdot z = 3d - z'$$

$$q_2 = +q \cdot z = 2d + z'$$



$$q_1' = -q \cdot z = -d - z'$$

$$q_2' = +q \cdot z = -2d + z'$$

$$q_3' = -q \cdot z = -3d - z'$$

The charge q at z' has two images: $q_1 = -q$ at $z = d - z'$ (reflection in upper plane) and $q_1' = -q$ at $z = -d - z'$ (reflection in lower plane).

But the image charges also have images!

All the images must occur outside V , the region in which we calculate the potential.

q_1 has an image $q_2' = +q$ at $z = -2d + z'$ and q_1' has an image $q_2 = +q$ at $z = 2d + z'$.

q_2 has an image q_3' and q_2' has an image q_3 .

And so on forever!

In V :

$$\Phi(\vec{r}) = q \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{(x-x')^2 + (y-y')^2 + [z - (-1)^n z' + nd]^2}}$$

for $-d/2 \leq z \leq +d/2$ only!

$|n|$ is the "image order"

$n=0$ is the real charge in V

$n < 0$ are images above the upper plate } not in V
 $n > 0$ are images below the lower plate }

If we set $q=1$ in the expression for $\Phi(\vec{r})$, we obtain the Dirichlet Green function for the two-plane geometry.

$$G_D(\vec{r}, \vec{r}') = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\sqrt{(x-x')^2 + (y-y')^2 + [z - (-1)^n z' + nd]^2}}$$

two planes

This can be used to solve any problem with Dirichlet boundary conditions between two plates.

For example: $\Phi(x, y, +d/2) = f_1(x, y)$ ← any function!

$\Phi(x, y, -d/2) = f_2(x, y)$ ←

A given charge distribution $g(\vec{r})$ in V .

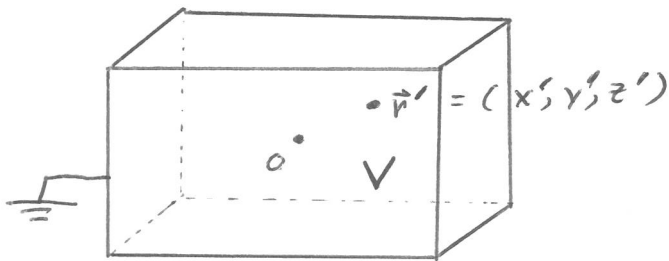
You can easily verify that

$$\nabla_{\vec{r}}^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r} - \vec{r}') \quad \text{in } V$$

and $G_D(\vec{r}, \vec{r}') = 0$ for $\vec{r} \text{ or } \vec{r}'$ on S

$$G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r})$$

It is now easy to generalize to the case of a grounded parallelepiped (box). The bounding planes are at $x = \pm \frac{a}{2}$, $y = \pm \frac{b}{2}$, $z = \pm \frac{c}{2}$.



$$G_D(\vec{r}, \vec{r}') = \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{(-1)^{l+m+n}}{\sqrt{[x - (-1)^l x' + la]^2 + [y - (-1)^m y' + mb]^2 + [z - (-1)^n z' + nc]^2}}$$

inside box

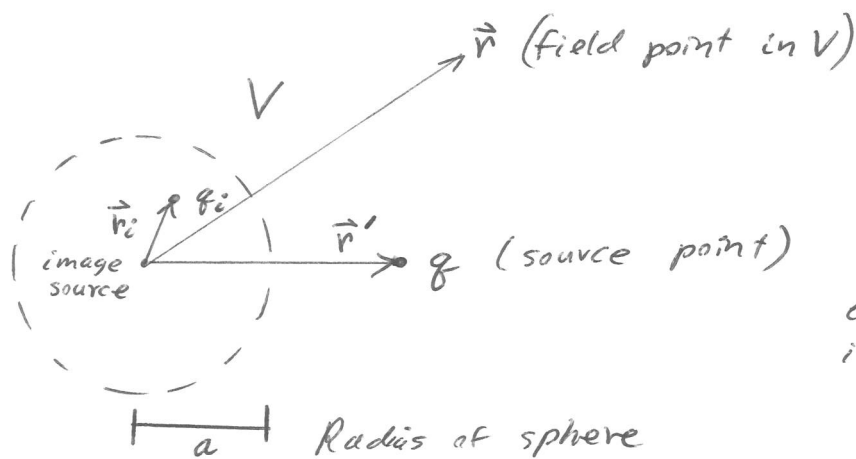
Now that we have the Dirichlet Green function, we can put any potentials on the six sides and any charges inside the box. Then the potential in V will be

$$\Phi(\vec{r}) = \int_V dV' \rho(\vec{r}') G_D(\vec{r}, \vec{r}') - \oint_S \frac{dS'}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$$

inside box

Of course, the potential outside V is zero.

Now consider a point charge q outside a grounded conducting sphere. We would like to find the potential outside the sphere. Any images must go inside the sphere. It is not obvious at all that the quartic surfaces of constant potential mentioned on page 7-1 can be made into a sphere! But it is true. We will remove the spherical conductor and put an image charge of magnitude q_i at position \vec{r}_i .



V is all space outside the sphere.

S is the surface of the sphere and the surface at infinity S_∞ .

Then in V , that is, for $|\vec{r}| > a$

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}'|} + \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

For a given q and \vec{r}' , we would like to find q_i and \vec{r}_i such that $\Phi(\vec{r}) = 0$ when $|\vec{r}| = a$. Define the vector \vec{a} to have length a and arbitrary direction. Then

$$0 = \frac{q}{|\vec{a} - \vec{r}'|} + \frac{q_i}{|\vec{a} - \vec{r}_i|}$$

So we see that q and q_i must have opposite sign. We write $q_i = -\lambda q$ where $\lambda > 0$.

$$|\vec{a} - \vec{r}_i| = \lambda |\vec{a} - \vec{r}'| \quad \text{square this}$$

$$(\vec{a} - \vec{r}_i) \cdot (\vec{a} - \vec{r}_i) = \lambda^2 (\vec{a} - \vec{r}') \cdot (\vec{a} - \vec{r}')$$

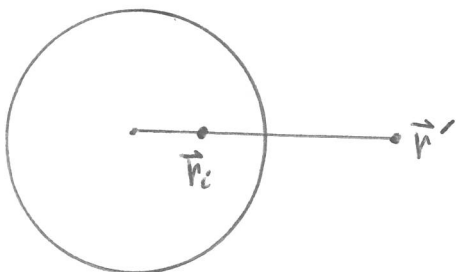
$$a^2 + r_i^2 - 2\vec{a} \cdot \vec{r}_i = \lambda^2 (a^2 + r'^2 - 2\vec{a} \cdot \vec{r}')$$

$$a^2(1 - \lambda^2) + \underline{2\vec{a} \cdot (\vec{r}_i - \lambda^2 \vec{r}')} + r_i^2 - \lambda^2 r'^2 = 0$$

Since this must hold for all directions of \vec{a} , the underlined term must vanish,

$$\vec{r}_i = \lambda^2 \vec{r}'$$

that is \vec{r}_i is along \vec{r}'
which is obvious from symmetry,



with the substitution $\vec{r}_i = \lambda^2 \vec{r}'$ we get

$$a^2(1-\lambda^2) + \lambda^4 r'^2 - \lambda^2 r'^2 = 0$$

or

$$(1-\lambda^2)(a^2 - \lambda^2 r'^2) = 0$$

which has two solutions $\begin{cases} \lambda = 1 \\ \lambda = a/r' \end{cases}$

For $\lambda = 1$, $\vec{r}_i = \vec{r}'$ and $q_i = -q$ which corresponds to two equal and opposite charges sitting on top of each other. This gives $\Phi = 0$ everywhere, not just on the spherical surface. Also this image charge is in V which is not allowed.

The real solution is therefore $\lambda = a/r'$.

The image charge is

$$q_i = -\frac{a}{r'} q$$

and it sits at

$$\vec{r}_i = \frac{a^2}{r'^2} \vec{r}'$$

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}'|} - \frac{a}{r'} \frac{q}{|\vec{r} - \frac{a^2}{r'^2} \vec{r}'|} \quad \text{for } |\vec{r}| > a$$

We have actually solved two problems here.

If $r' < a$ then $r_i > a$ and

If $r' > a$ then $r_i < a$. Thus if the real

charge is inside the sphere, the image charge is outside and vice-versa.

If we set $q=1$ in the potential we get the Dirichlet Green function:

$$G_D(\vec{r}, \vec{r}') = \frac{1}{|\vec{r} - \vec{r}'|} - \frac{a}{r'} \frac{1}{|\vec{r} - \frac{a^2}{r'^2} \vec{r}'|}$$

If \vec{r} and \vec{r}' are inside the sphere, this is the Dirichlet Green function for the interior of the sphere. But if \vec{r} and \vec{r}' are outside the sphere, this is the Dirichlet Green function for the exterior! Two solutions for the price of one.

If we are interested in the Force that the sphere exerts on the charge q , there are two ways to calculate it. The long way is ~~to~~ take the derivative of the potential at the surface of the sphere to get the electric field and charge density, then integrate all the infinitesimal

Coulomb forces between the charges to get the total force. The easy way is simply to consider the single Coulomb force between the real charge and the image charge. The two answers will of course be identical.

$$\begin{aligned} \vec{F}_{\text{sphere on } q} &= q q_i \frac{(\vec{r}' - \vec{r}_i)}{|\vec{r}' - \vec{r}_i|^3} = q \left(-\frac{a}{r'} q\right) \frac{(\vec{r}' - \frac{a^2}{r'^2} \vec{r}')}{|\vec{r}' - \frac{a^2}{r'^2} \vec{r}'|^3} \\ &= -q^2 \frac{a \vec{r}'}{(r'^2 - a^2)^2} \end{aligned}$$

Notice that this force is always attractive, no matter what the sign of the real charge is.

A positive charge is attracted to the grounded sphere as much as is a negative charge!

The Green function does not look symmetric

$$\begin{aligned} G_0(\vec{r}, \vec{r}') &= \frac{1}{|\vec{r} - \vec{r}'|} - \frac{a}{r'} \frac{1}{|\vec{r} - \frac{a^2}{r'^2} \vec{r}'|} \quad \text{but it is} \\ &= \frac{1}{\sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + r'^2}} - \frac{a}{\sqrt{r^2 r'^2 - 2a^2 \vec{r} \cdot \vec{r}' + a^4}} \end{aligned}$$

now the symmetry is manifest.

Now consider a slightly more complicated problem:

A point charge outside a spherical conductor held at potential Φ_0 . Find $\Phi(\vec{r})$ outside the sphere.

We might guess the following:

$$\Phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} - \frac{a}{r'} \frac{q}{|\vec{r}-\frac{a^2}{r'^2}\vec{r}'|} + \Phi_0$$

This satisfies Poisson's Equation outside the sphere

$$\nabla^2 \Phi(\vec{r}) = -4\pi q \delta^3(\vec{r}-\vec{r}') = -4\pi q(\vec{r})$$

and has the right boundary condition at $r=a$, on the surface of the sphere

$$\Phi(r=a) = \Phi_0 \quad \checkmark$$

but it violates the boundary condition on the surface at infinity, which is part of S ,

$$\Phi(r \rightarrow \infty) = \Phi_0 \neq 0 \quad \times$$

Let's try adding another image charge to the interior of the sphere. If we put an image charge at the origin, the image of the image will be at infinity and this is fine. If we put an image anywhere but the origin, the image of the image will be in V and that is not allowed.

So we try a charge of magnitude $q_0 = \Phi_0 a$ at the origin, then outside the sphere:

$$\Phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{r}'|} - \frac{a}{r'} \frac{q}{|\vec{r} - \frac{a^2}{r'^2} \vec{r}'|} + \frac{\Phi_0 a}{r}$$

Now we satisfy Poisson's Equation and all Dirichlet boundary conditions $\Phi(r \rightarrow \infty) = 0$.

As an exercise, let's calculate the charge density induced on an infinite grounded conducting plane by a point charge a distance $\frac{d}{2}$ away.

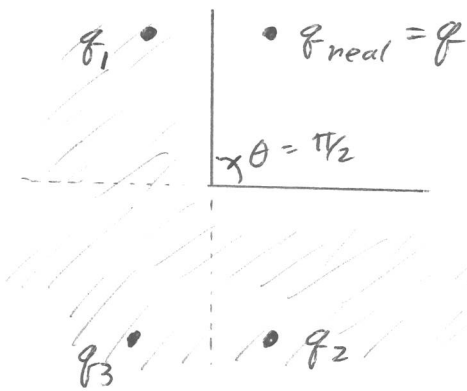
$$\Phi(x, y, z) = \frac{q}{\sqrt{x^2 + y^2 + (z - \frac{d}{2})^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z + \frac{d}{2})^2}}$$

The surface charge density is:

$$\begin{aligned} \sigma(x, y) &= \frac{1}{4\pi} \hat{n}_0 \cdot \vec{E}(x, y, 0) = \frac{1}{4\pi} \hat{e}_z \cdot \vec{E}(x, y, 0) = \frac{1}{4\pi} E_z(x, y, 0) \\ &= -\frac{1}{4\pi} \left[\frac{\partial}{\partial z} \Phi(x, y, z) \right]_{z=0} \\ &= \frac{-q}{4\pi} \left\{ -\left(z - \frac{d}{2}\right) \left[x^2 + y^2 + \left(z - \frac{d}{2}\right)^2\right]^{-3/2} + \left(z + \frac{d}{2}\right) \left[x^2 + y^2 + \left(z + \frac{d}{2}\right)^2\right]^{-3/2} \right\} \Bigg|_{z=0} \\ &= -\frac{qd}{4\pi} \left[x^2 + y^2 + \frac{d^2}{4} \right]^{-3/2} \end{aligned}$$

The total charge induced is of course $-q$.

Other geometries which lend themselves to solutions by the method of images are:



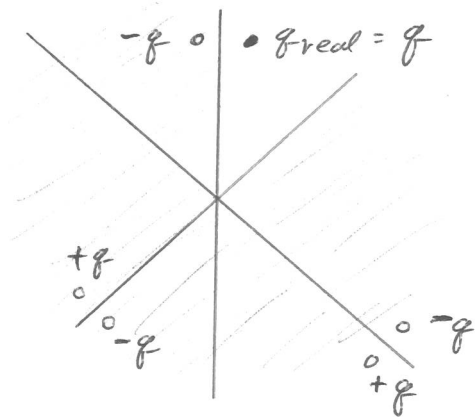
i) semi-infinite planes at right angles (3 images)

$$q_1 = -q$$

$$q_2 = -q$$

$$q_3 = +q$$

ii) semi-infinite planes intersecting at $\theta = \frac{\pi}{n}$ ($2n-1$) images



iii) 3 intersecting planes at right angles (like the corner in a room).

Images and images of images must never appear in the volume of interest V .