The magnetic multipole moment tensors that appear in the expansion of the magnetic scalar potential

$$\Phi_m(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{4\pi}{2\ell+1} \mathfrak{M}_{\ell m} \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta,\varphi)$$

also appear in a multipole expansion of the vector potential $\vec{A}(\vec{r})$, but it is very involved to show how they appear. We will not go through the computation.

<u>Interaction of currents with an external magnetic field</u>

Up to now we have been concerned with how a current gives rise to a field. We now consider the problem of an established external magnetic field acting on a current. We will assume that the field produced by this current is negligible compared to the external field. First we consider the <u>force</u> on a system of moving charges:

$$\vec{F}_{mag} = \sum_{i=1}^{N} q_i \frac{\vec{v}_i}{c} \times \vec{B}_{ext}(\vec{r}_i)$$

go to continuum limit

$$\rightarrow \int dV\, \rho(\vec{r}) \frac{\vec{v}(\vec{r})}{c} \times \vec{B}_{ext}(\vec{r})$$

$$= \int dV\, \frac{\vec{J}(\vec{r})}{c} \times \vec{B}_{ext}(\vec{r})$$

Thus $\frac{1}{c}\vec{J}(\vec{r}) \times \vec{B}_{ext}(\vec{r})$ is a magnetic <u>force</u> <u>density</u>. If the current system is confined to a small region, it seems reasonable to make a Taylor series expansion of $\vec{B}_{ext}$ about some point in the midst of the current distribution. The i^{th} component of the force is

$$F_i = \frac{1}{c} \sum_j \sum_k \epsilon_{ijk} \int dV \; J_j(\vec{r}) \, B_k(\vec{r})$$

Then we expand $B_k(\vec{r})$ about the origin (the "center" of the current system).

$$B_k(\vec{r}) = B_k(0) + \sum_\ell \left(\frac{\partial B_k}{\partial x_\ell} \right) \Bigg|_{\vec{r}=0} x_\ell + \cdots$$

and thus

$$F_i = \sum_j \sum_k \epsilon_{ijk} \left[\int dV \frac{J_j(\vec{r})}{c} B_k(0) + \sum_\ell \int dV \, x_\ell \frac{J_j(\vec{r})}{c} \frac{\partial B_k}{\partial x_\ell}\bigg|_0 + \cdots \right]$$

In the first term $B_k(0)$ is a constant, so we consider

$$\int dV \; \vec{J}(\vec{r})$$

Since charge is conserved $\frac{\partial \rho(\vec{r})}{\partial t} = 0$ and the continuity equation implies

$$\vec{\nabla} \cdot \vec{J}(\vec{r}) = 0$$

Since the divergence of $\vec{J}(\vec{r})$ vanishes, we may write

$$\vec{J}(\vec{r}) = \vec{\nabla} \times \vec{\Psi}(\vec{r}) \qquad \text{since } \vec{\nabla} \cdot \vec{\nabla} \times \overrightarrow{\text{Anything}} = 0$$

where $\vec{\Psi}(\vec{r})$ is a vector field that vanishes when $\vec{J}(\vec{r})$ vanishes. But then

$$\int_V dV\, \vec{J}(\vec{r}) = \int_V dV\, \vec{\nabla} \times \vec{\Psi}(\vec{r}) = \oint_S dS\, \hat{n} \times \vec{\Psi}(\vec{r})$$

and we can choose the surface in the last integral far outside the current distribution where $\vec{\Psi}(\vec{r}) = 0$.

Thus $\int dV\, \vec{J}(\vec{r}) = 0$

The second term in the expression for the force involves

$$\frac{1}{c} \int dV\, x_\ell\, J_j(\vec{r})$$

We write this as a symmetric piece plus an anti-symmetric piece: $\quad A = \frac{1}{2}(A+B) + \frac{1}{2}(A-B)$

$$\frac{1}{c} \int dV\, x_\ell\, J_j(\vec{r}) = \frac{1}{2c} \int dV \left[x_\ell J_j - x_j J_\ell\right] + \frac{1}{2c} \int dV \left[x_\ell J_j + x_j J_\ell\right]$$

It is a homework assignment to show that the second term vanishes

$$\frac{1}{2c} \int dV \left[x_\ell J_j + x_j J_\ell\right] = 0$$

Then $\frac{1}{c}\int dV\, x_\ell\, J_j(\vec{r}) = \frac{1}{2c}\int dV\left[x_\ell\, J_j(\vec{r}) - x_j\, J_\ell(\vec{r})\right]$

$$= \frac{1}{2c}\sum_n \epsilon_{n\ell j}\int dV\,\left[\vec{r}\times\vec{J}(\vec{r})\right]_n = \sum_n \epsilon_{n\ell j}\, m_n$$

↑ dipole moment

So the i^{th} component of the force on the current distribution is

$$F_i = \sum_j\sum_k\sum_\ell\sum_n \epsilon_{ijk}\,\epsilon_{n\ell j}\, m_n\, \left.\frac{\partial B_k}{\partial x_\ell}\right|_{\vec{r}=0}$$

But $\sum_j \epsilon_{ijk}\,\epsilon_{n\ell j} = \delta_{i\ell}\,\delta_{kn} - \delta_{in}\,\delta_{k\ell}$

$$\therefore\; F_i = -m_i\left[\vec{\nabla}\cdot\vec{B}(\vec{r})\right]_{\vec{r}=0} + \left[\frac{\partial}{\partial x_i}(\vec{m}\cdot\vec{B})\right]_{\vec{r}=0}$$

Since $\vec{\nabla}\cdot\vec{B}(\vec{r}) = 0$ everywhere, the first term vanishes.

Next, we move the origin of the current distribution from O to some general point $\vec{r}$.

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$$

This expression is <u>exact</u> for point magnetic dipoles or for fields which have only a first derivative.

In general $\vec{\nabla}(\vec{m}\cdot\vec{B})$ is only the first term in an expansion. If $\underline{a}$ is roughly the radius of the current distribution and $\underline{\lambda}$ is the length scale over which $\vec{B}$ changes appreciably, then

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) \quad \text{is} \quad \mathcal{O}\left(\frac{a}{\lambda}\right)$$

the next correction is $\mathcal{O}\left(\frac{a^2}{\lambda^2}\right)$ and hence will be negligible if $a \ll \lambda$.

A current distribution will also experience a <u>torque</u> in addition to a force. The torque density is

$$\vec{r} \times \left[\frac{\vec{J}(\vec{r})}{c} \times \vec{B}(\vec{r})\right]$$

and the torque is

$$\vec{\tau} = \int \frac{dV}{c}\, \vec{r} \times \left[\vec{J}(\vec{r}) \times \vec{B}(\vec{r})\right] = \int \frac{dV}{c}\left[(\vec{r}\cdot\vec{B})\vec{J} - (\vec{r}\cdot\vec{J})\vec{B}\right]$$

Again, we expand $\vec{B}$ about the center of the current distribution. This time, we keep only the first term, $\vec{B}(0)$.

$$\tau_i = \sum_\ell \int \frac{dV}{c}\, x_\ell J_i B_\ell(0) - \sum_\ell \int \frac{dV}{c}\, x_\ell J_\ell B_i(0)$$

Previously, we found that

$$\int \frac{dV}{c} x_\ell J_i = \sum_k \epsilon_{\ell i k} m_k$$

and hence

$$\int \frac{dV}{c} x_\ell J_\ell = \sum_k \epsilon_{\ell \ell k} m_k = 0 \qquad \text{since } \epsilon \text{ is totally antisymmetric}$$

$$\tau_i = \sum_\ell \sum_k \epsilon_{\ell i k} m_k B_\ell(0) = \sum_\ell \sum_k \epsilon_{i k \ell} m_k B_\ell(0)$$

since $\epsilon_{123} = \epsilon_{312} = \epsilon_{231}$

In full vector form, this is

$$\vec{\tau}_0 = \vec{m} \times \vec{B}(0)$$

where the torque is evaluated about an axis through O.

If we move the origin to a general point P, then

$$\vec{\tau}_P = m \times \vec{B}(P)$$

Again, this is exact for a point dipole and more generally is only the leading term $\left[\mathcal{O}\left(\frac{a}{\lambda}\right)\right]$ in an expansion.

Both the force and the torque can be determined by assuming a <u>potential energy</u> for the magnetic dipole in an external magnetic field:

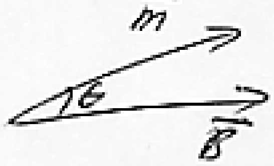
$$U = -\vec{m} \cdot \vec{B}$$

By taking a length as the generalized coordinate, we obtain the force:

$$\vec{F} = -\vec{\nabla} U = +\vec{\nabla}(\vec{m} \cdot \vec{B})$$

and by taking an angle as the generalized coordinate we obtain the torque:

$$\vec{\tau} = -\frac{\partial}{\partial \theta} U \, \hat{e}_\theta$$



$$\vec{\tau} = \frac{\partial}{\partial \theta}(mB\cos\theta)\,\hat{e}_\theta = -mB\sin\theta\,\hat{e}_\theta = \vec{m} \times \vec{B}$$

At first glance, it is puzzling that there can be an energy associated with magnetic forces since the magnetic force on a charge is $q\frac{\vec{v}}{c} \times \vec{B}$ and thus is always perpendicular to the velocity of the particle, and hence the infinitesimal displacement that contributes to the work. Magnetic forces do no work. However, when the currents are pushed through an inhomogeneous magnetic field, the field will appear to change in time

(in the frame of reference of the current system).
We will see that an electric field will then appear and this electric field can do work. If the currents are to be maintained in steady state, then external work must be done on them. This is the source of the energy

$U = -\vec{m} \cdot \vec{B}$.

In a wire, the force that pushes charges along the wire to form a current when the wire is moved through an inhomogeneous magnetic field is called the electromotive force, emf.

End lecture # 21