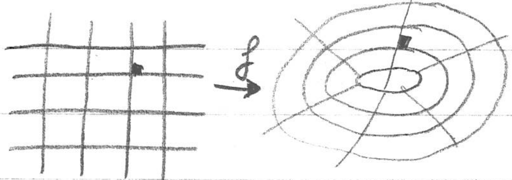
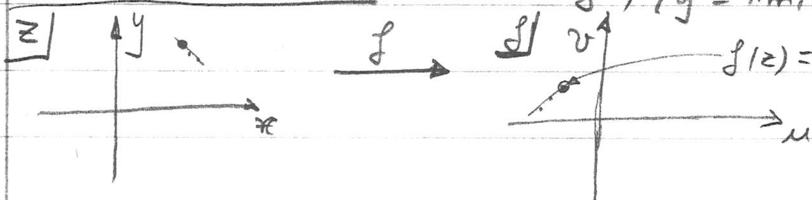
Conformal Mapping

Conformal = mapping preserves angles



(Pb) simple boundary $\xrightarrow{f}$ non-trivial boundary
know solution

Complex coordinate: $z = x + iy$; $\begin{cases} x = Re(z) \\ y = Im(z) \end{cases}$



$f(z) = u(x,y) + i v(x,y)$

$u = Re(F)$
$v = Im(F)$

f analytic if the derivative of f : ① exists ② is unique

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$; no fixed dir. for Δz → you need to do all directions

Real direction: $$\frac{df}{dz} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Imaginary direction : $$\frac{df}{dz} = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y) + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

But derivatives must be the same in ∀ direction →

Cauchy - Riemann Equations: $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}}$, $\boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$ $\Big\}$ f is analytic

Ex: $f(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i(2xy) = u(x,y) + i v(x,y)$

$\frac{\partial u}{\partial x} = 2x$ $\quad \frac{\partial v}{\partial y} = 2x$; $\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$; $\frac{\partial u}{\partial y} = -2y$

if f is analytic → curves of constant u are ⊥ to curves of const. v

$$\left.\begin{array}{l}\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 v}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\partial^2 v}{\partial x \partial y}\end{array}\right\}$$

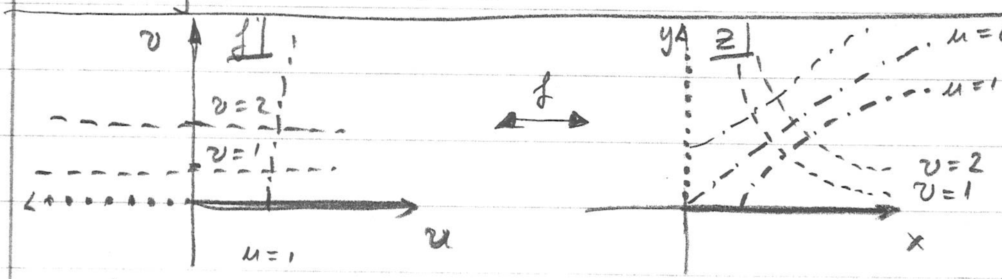
for v well behaved $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow \boxed{\nabla^2 u(x,y) = 0}$$

Similar approach $\rightarrow \boxed{\nabla^2 v(x,y) = 0}$

u, v called conjugate harmonic functions

$z = x + iy = re^{i\theta} \rightarrow z^2 = r^2 e^{i2\theta}$

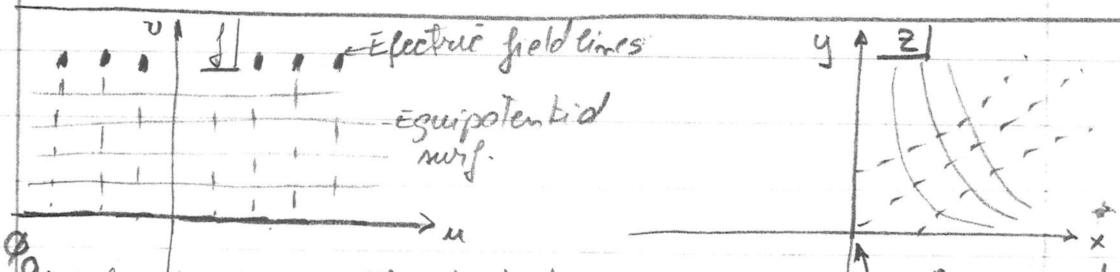


$u + iv = (x^2 - y^2) + i(2xy)$

$v = 1 \rightarrow 2xy = 1$

$u = 1 \rightarrow x^2 - y^2 = 1$

$u = 0 \rightarrow x^2 - y^2 = 0$



ϕ_0 infinite plane with potential ϕ_0

angles are not preserved!!
→ Exception $\boxed{f'(0) = 0}$ or singularities

$\phi(u,v) = \phi_0 - E_{0,v} v$ (const) not analytic → Build:

could be either real of imaginary part of H (your choice)

$H(f) = H(u,v) = -E_0 u + i[\phi_0 - E_0 v] = i\phi_0 - E_0 f \rightarrow H(f)$ analytic

prove using Cauchy Riemann rel

$E_u = -\dfrac{\partial \phi}{\partial u} = 0$

$f = z^2$ conformal mapping that folds the plane into 2d └

$E_v = -\dfrac{\partial \phi}{\partial v} = E_0$

$H(z^2) = i\phi_0 - E_0 z^2 = i\phi_0 - E_0(x + iy)^2 = i\phi_0 - E_0(x^2 - y^2) - E_0 i(2xy)$

steps ① wrote potential in ζ space
② write H analytic (in wich the pot is either Im or Re part
③ use conformal mapping ($\zeta = z^2$)
④

$\zeta = z^2 \rightarrow$ $\llcorner$

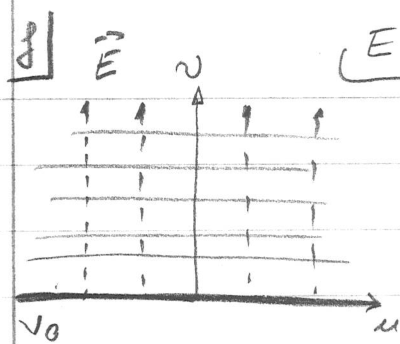
$\zeta = z^4 \rightarrow$ $\angle^{45}$

$\zeta = z^{\frac{180}{\theta}} \rightarrow$ $\angle\theta$

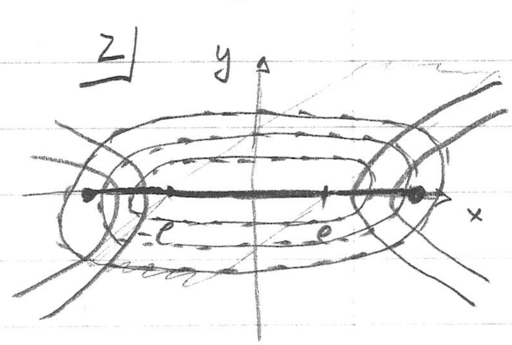
Lecture 17

03/31/05

EM



$z = \ell \sin(f)$



$f = u + iv$ $\qquad\qquad$ $z = x + iy$

$$z = \ell \sin f = \ell \sin(u+iv) = \ell\left(\frac{e^{i(u+iv)} - e^{-i(u+iv)}}{2i}\right) =$$

$$= \frac{\ell}{2i}\left(e^{iu}\, e^{-v} - e^{-iu}\, e^{v}\right) = \frac{\ell}{2i}\left[(\cos u + i \sin u)e^{-v} - (\cos u - i \sin u)e^{v}\right]$$

$$= \frac{\ell}{2i}\left[\cos u\,(e^{-v} - e^{v}) + i \sin u\,(e^{-v} + e^{v})\right]$$

$$= \ell\left[\cos u\left(-\frac{\sinh v}{i}\right) + \sin u \cosh v\right]$$

$$= i\ell \cos u \sinh v + \ell \sin u \cosh v$$

$$= x + iy \rightarrow \begin{cases} x = \ell \sin u \cosh v \\ y = \ell \cos u \sinh v \end{cases}$$

$\forall u \xrightarrow{\text{const}} \cos^2 u + \sin^2 u = 1$

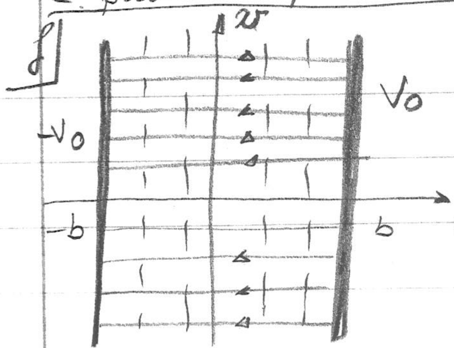
$$\rightarrow \left[\frac{y}{\ell} \sinh v\right]^2 + \left[\frac{x}{\ell} \cosh v\right] = 1 \qquad \text{elipse equations}$$

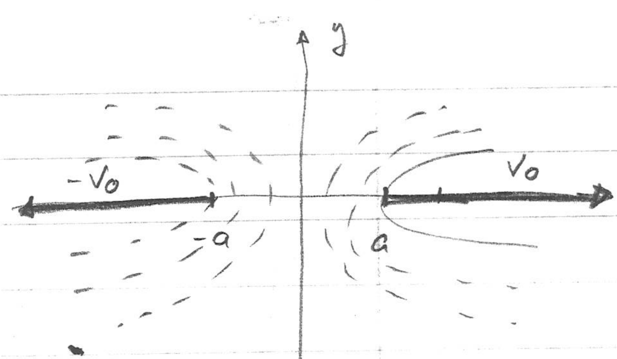
$\rightarrow$ lines of const v in $\underline{f|}$ $\rightarrow$ elipses in $\underline{z|}$

$\forall v = \text{const} \rightarrow \cosh^2(v) - \sinh^2(v) = 1 \rightarrow$

$$\left[\frac{x}{\ell \sin u}\right]^2 - \left[\frac{y}{\ell \cos u}\right]^2 = 1 \qquad \text{hiperbolo eq.}$$

2 plane capacitators $\longrightarrow$ 2 semi∞ potential planes





$$f = \frac{2b}{\pi} \arcsin\left(\frac{z}{a}\right) \qquad z = a \sin\left(\frac{\pi f}{2b}\right)$$

$$z = a \sin\left[\frac{\pi(u + iv)}{2b}\right] = \dots = \underbrace{a \sin\left(\frac{\pi u}{2b}\right) \cosh\left(\frac{\pi v}{2b}\right)}_{x} + i \underbrace{a \cos\left(\frac{\pi u}{2b}\right) \sinh\left(\frac{\pi v}{2b}\right)}_{y}$$

$$\rightarrow \begin{cases} x = a \sin\left(\frac{\pi u}{2b}\right) \cosh\left(\frac{\pi v}{2b}\right) \\ y = a \cos\left(\frac{\pi u}{2b}\right) \sinh\left(\frac{\pi v}{2b}\right) \end{cases}$$

equipotentia lines $\rightarrow$ lines of const u

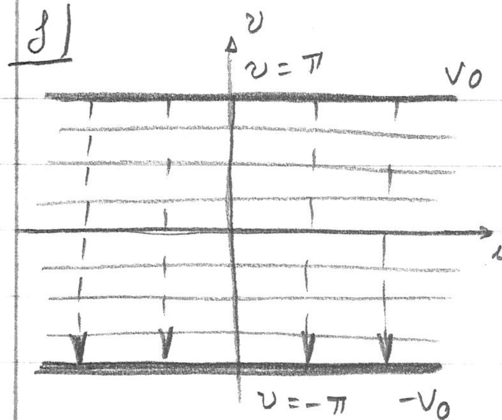
$$\cosh^2\left(\frac{\pi v}{2b}\right) - \sinh^2\left(\frac{\pi v}{2b}\right) = 1$$

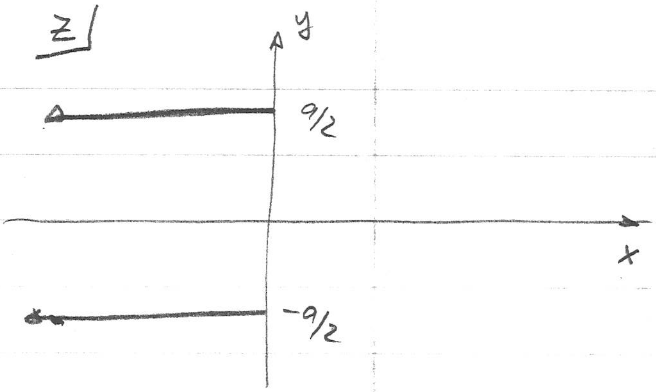
$$\left[\frac{x}{a \sin\left(\frac{\pi u}{2b}\right)}\right]^2 - \left[\frac{y}{a \cos\left(\frac{\pi u}{2b}\right)}\right]^2 = 1 \quad \text{hyperbolas}$$

HW: Find the field lines !!

∞ parallel plate capacitor

semi ∞ plate capacitor





$$\boxed{z=\frac{a}{2\pi}(1+\zeta+e^{\zeta})}$$

① (HW) prove it is analytic
C-R eq.

$$z=\frac{a}{2\pi}\left[1+(u+iv)+e^{u+iv}\right]=\frac{a}{2\pi}\left[1+u+iv+e^{u}\cdot\underbrace{e^{iv}}_{\cos v+i\sin v}\right)$$

$$=x+iy \rightarrow \begin{cases} x=\frac{a}{2\pi}(1+u+e^{u}\cos v) \\ y=\frac{a}{2\pi}(v+e^{u}\sin v) \end{cases}$$

② What are the curves describing electric field lines E

(see notes!)
(start and end on metal)
(on one on the other one)

due on 8 April.

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

If these conditions are met, f is analytic

$$f(z) = z^2 = (x+iy)^2$$
$$= (x^2-y^2) + i(2xy)$$
$$= u(x,y) + iv(x,y)$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2-y^2) = 2x \qquad \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x \qquad \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2-y^2) = -2y$$

Take second derivatives, and add:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

if fuction is well behaved, may interchange order of derivatives

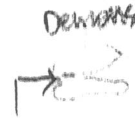
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 u(x,y) = 0$$
$$\nabla^2 v(x,y) = 0$$

De Moivre's Theorem:

$$z = re^{i\theta}$$
$$z^2 = r^2 e^{2i\theta}$$

Demonstration: Parking Garage



"branch cut" is the intersection

$$H(f) = H(u,v)$$

$$
\begin{aligned}
H(z^2) &= i\Phi_0 - E_0 z^2 \\
&= i\Phi_0 - E_0 (x+iy)^2 \\
&= i\Phi_0 - E_0 (x^2-y^2) - E_0 (i2xy)
\end{aligned}
$$

$180^\circ \rightarrow 90^\circ \qquad f = z^2$

$180^\circ \rightarrow 45^\circ \qquad f = z^4$

$180^\circ \rightarrow \theta \qquad f = z^{\frac{180}{\theta}}$

$$V(r<a) = \frac{\sigma_0 r}{2\varepsilon_0}\cos\varphi$$

and
$$V(r>a) = \frac{\sigma_0 a^2}{2\varepsilon_0 r}\cos\varphi$$

5.2.3 Conformal Mappings

The idea behind conformal mappings is to take the solution to a very simple boundary condition problem and then to fold, stretch, or otherwise deform the boundary by a *conformal* mapping to match the boundary for a more complicated problem of interest. The same mapping that changes the boundary will also deform the field lines and constant potential lines of the simple problem to those of the more complicated problem. As the use of complex functions is central to the technique, we begin with a consideration of complex functions of a complex variable $z = x + iy$.

Generally, a complex function $f(z)$ may be written as the sum of a real part and an imaginary part, $f(z) = u(x,y) + i\,v(x,y)$. This means that for a point $x + iy$ in the (complex) z plane we can find a corresponding point $u + iv$ in the *image plane* (or, more briefly, the f plane) defined by f. The function f may be said to *map* the point (x, iy) to (u, iv). A series of points in the z plane will be mapped by f to a series of points in the f plane. If the function f is well behaved, adjacent points in the z plane are mapped to adjacent points in the f plane (i.e., a line in the z plane is mapped to a line in the f plane). It is such well-behaved complex functions or mappings that we will consider.

A function $f(z)$ is said to be *analytic* (the terms *regular* or *holomorphic* are also used) at a point z_0 if its derivative, defined by

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \tag{5-11}$$

exists (and has a unique value) in some neighborhood of z_0. A moment's reflection will show this to be a considerably stronger condition than the equivalent for functions of real variables because Δz can point in any direction in the z plane. The direction of taking this limit is immaterial for an analytic function. To investigate the consequence of this property, let us take the derivative first in the real direction and then in the imaginary direction.

Taking Δz along the real axis results in

$$\begin{aligned}\frac{df}{dz} &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y) - u(x,y) + i\,[v(x+\Delta x,y) - v(x,y)]}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\end{aligned} \tag{5-12}$$

while taking the derivative in the imaginary direction results in

$$\begin{aligned}\frac{df}{dz} &= \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y) + i[v(x,y+\Delta y) - v(x,y)]}{i\,\Delta y} \\ &= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\end{aligned} \tag{5-13}$$

Comparing the two expressions (5-12) and (5-13), we find that the real and imaginary parts of an analytic function f are not arbitrary but must be must be related by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5-14}$$

These two equalities (5-14) are known as the *Cauchy-Riemann equations.* The validity of.the Cauchy-Riemann equations is both a necessary and sufficient condition for f to be analytic.

To illustrate these ideas, we consider the function $f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2ixy$. Then $u = x^2 - y^2$ and $v = 2xy$. The derivatives are easily obtained

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x; \qquad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

in accordance with the Cauchy-Riemann equations.

The Cauchy-Riemann equations can be differentiated to get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}$$

Since the order of differentiation should be immaterial, these can be added to give

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0$$

In similar fashion

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y\,\partial x} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x\,\partial y} \quad \Rightarrow \quad \nabla^2 v = 0$$

The functions u and v each solve Laplace's equation. Any analytic function therefore supplies two solutions to Laplace's equation, suggesting that we might look for the solutions of static potential problems among analytic functions. The functions u and v are known as *conjugate harmonic functions*. It is easily verified that the curves u = constant and v = constant are perpendicular to one another, suggesting that if one were to represent curves of constant potential, then the other would represent electric field lines.

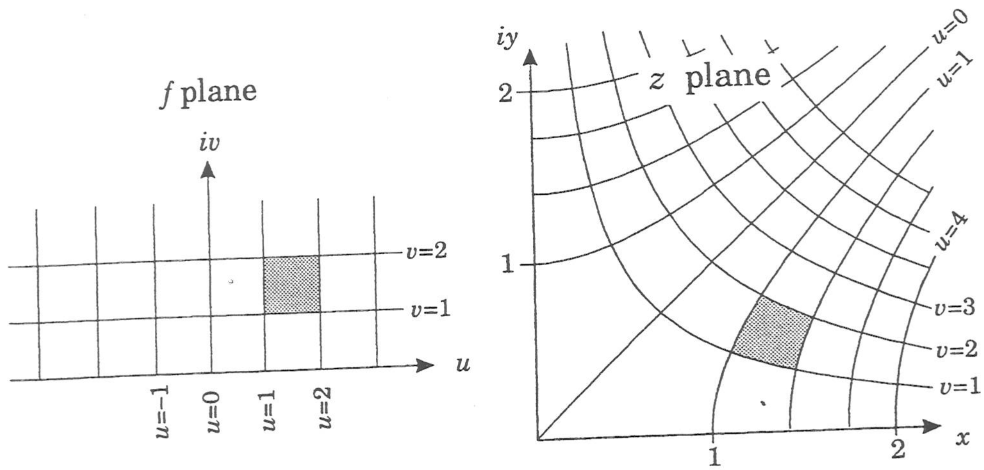


Figure 5.7 The image of the rectangular grid in the f plane under the mapping $z = \sqrt{f}$ is shown on the right in the z plane.

To return now to the notion of conformal mappings,† let us consider the mapping produced by the analytic function $f(z) = z^2$ (Figure 5.7). The inverse mapping, $z = \sqrt{f}$, is analytic everywhere except at $z = 0$. The image of the line $v = 0$ in the z plane is easily found: for $u > 0$, $z = \sqrt{u}$ produces an image line along the positive x axis, while for $u < 0$, $z = \sqrt{u} = i\sqrt{-u}$ produces an image line along the positive y axis of the z plane. The image of $v = 1$ is easily found from $(u + iv) = x^2 - y^2 + 2ixy$, implying that $2xy = 1$. Similarly, the line $u = c$ has image $x^2 - y^2 = c$.

Let us now consider the potential above an infinite flat conducting plate with potential $V = V_0$, lying along the u axis. Above the plate, the potential will be of the form $V = V_0 - av$. (A second plate at a different potential, parallel to the first would be required to determine the constant a.) In a more general problem, V would be a function of both u and v. V may generally be considered to be the imaginary (or alternatively, real) part of an analytic function

$$\Phi(u, v) = U(u, v) + iV(u, v) = \Phi(f) \tag{5-15}$$

For this example, taking V to be the imaginary part of an analytic function Φ, $\Phi(u,v) = iV_0 - af = -au + i(V_0 - av)$. The corresponding electric field has components $E_v = a$ and $E_u = 0$.

---

† The mappings are called conformal because they preserve the angles between intersecting lines except when singular or having zero derivative. To see this, we consider two adjacent points z_0 and z on a line segment that makes angle φ with the x axis at z_0. Then $z - z_0$ may be written in polar form as $re^{i\varphi}$. The image of the segment $f(z) - f(z_0)$ can be expressed to first order as $f'(z_0)(z - z_0)$. If we write each term in polar form, Δf takes the form $Re^{i\theta} = ae^{i\alpha}re^{i\varphi}$. Thus the image line segment running from $f(z_0)$ to $f(z)$ makes angle $\theta = \alpha + \varphi$ with the u axis. This means that any line passing through z_0 is rotated through angle α in the mapping to the f plane providing that the derivative $f'(z_0)$ exists and that a unique polar angle α can be assigned to it. When f' is zero as for $f(z) = z^2$ at $z = 0$, conformality fails.

Now the mapping that takes the grid lines from the f plane to the z plane also takes Φ to the z plane. Expressed in terms of x and y

$$\Phi[f(z)] = iV_0 - az^2 = a(y^2 - x^2) + i(V_0 - 2axy)$$

Thus, in the z plane, $U(x,y) = a(y^2 - x^2)$ and $V(x,y) = V_0 - 2axy$. The latter is the potential produced by two conducting plates at potential V_0 intersecting at right angles. The field lines produced by v = constant in the u-v plane are now obtained from $u = y^2 - x^2$ = constant, while the constant potential lines are produced by $v = 2xy$ = constant.

A second example is offered by the mapping $z = \ell \sin f$, which maps the entire u axis onto a finite line segment of length 2ℓ in the z plane. We express z in terms of u and v by expanding $\sin f$ as

$$\sin f = \frac{e^{if} - e^{-if}}{2i}$$

to obtain

$$\begin{aligned}\sin f &= \frac{e^{i(u+iv)} - e^{-i(u+iv)}}{2i} = \frac{e^{-v}e^{iu} - e^{v}e^{-iu}}{2i} \\ &= \frac{e^{-v}(\cos u + i\sin u) - e^{v}(\cos u - i\sin u)}{2i} \\ &= i\sinh v \cdot \cos u + \cosh v \cdot \sin u\end{aligned}$$

Thus $x = \ell \cosh v \cdot \sin u$ and $y = \ell \sinh v \cdot \cos u$.

We again take for the potential above the $v = 0$ "plane" the imaginary part of $\Phi(u,v) = iV_0 - af$. It is somewhat awkward, however, to find Φ in terms of x and y directly by substituting for f in the expression for Φ. Fortunately this is not really necessary. For a given point (u, v) in the f plane (where V is readily calculated), the corresponding (x, y) value is readily found. In particular, the equipotentials at v = constant are easily obtained, for

$$\frac{x^2}{\ell^2\cosh^2 v} + \frac{y^2}{\ell^2\sinh^2 v} = 1 \tag{5-16}$$

In other words, the equipotentials around the flat strip are ellipses of major axis $\ell \cosh v$ and minor axis $\ell \sinh v$. At large distances $\cosh v$ and $\sinh v$ both tend to $\frac{1}{2}e^v$, meaning the equipotentials tend to circles. The potential and field lines are shown in Figure 5.8. (This problem can also be solved using separation of variables in elliptical coordinates.)

For a last example of somewhat more interest, let us consider the transformation

$$z = \frac{a}{2\pi}\left(1 + f + e^f\right) \tag{5-17}$$

If f is real, then z is also real, and, as u varies from $-\infty$ to $+\infty$, x

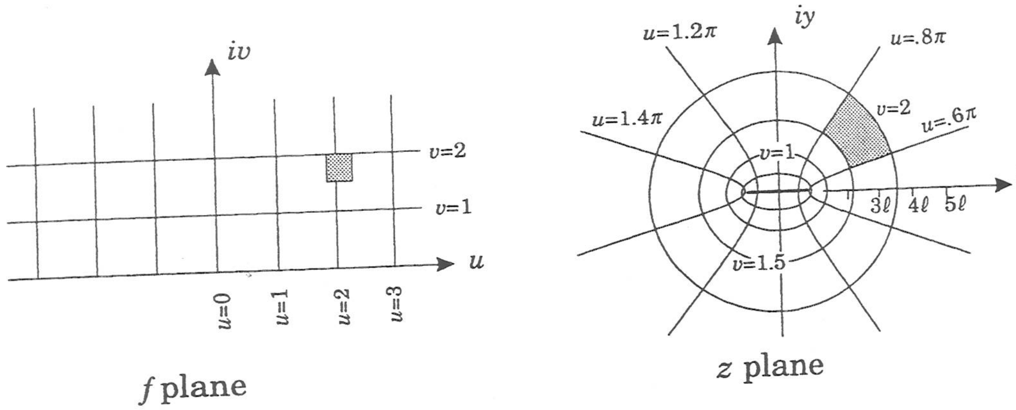


Figure 5.8 The figure on the left is the mapping of the regular grid on the right produced by the function $z = \ell \sin f$.

will take on values from $-\infty$ to $+\infty$. The real axis maps (albeit non-linearly) onto itself. Next consider the horizontal line $f = u + i\pi$. Substituting this into the expression for z gives

$$z = \frac{a}{2\pi}\left(1 + u + i\pi + e^{u+i\pi}\right) = \frac{a}{2\pi}\left(1 + u + i\pi - e^{u}\right)$$

We conclude that

$$x = \frac{a}{2\pi}\left(1 + u - e^{u}\right) \quad \text{and} \quad y = \frac{a}{2}$$

For u large and negative, $x = au/2\pi$ increases to 0 as u goes to zero. As u passes zero, e^u exceeds $1 + u$ and x retraces the negative x axis from 0 to $-\infty$. The line in the f plane folds back on itself at $u = 0$ on being mapped to the z plane. Similarly, the line at $v = -\pi$ maps to the half-infinite line at $y = -a/2$ with negative x. If we let the lines at $\pm i\pi$ in the f plane be the constant potential plates of the infinite parallel plate capacitor of Figure 5.9, the mapping carries this potential to that of the semi-infinite capacitor on the right. We can with this mapping find the fringing fields and potentials near the edge of a finite parallel plate

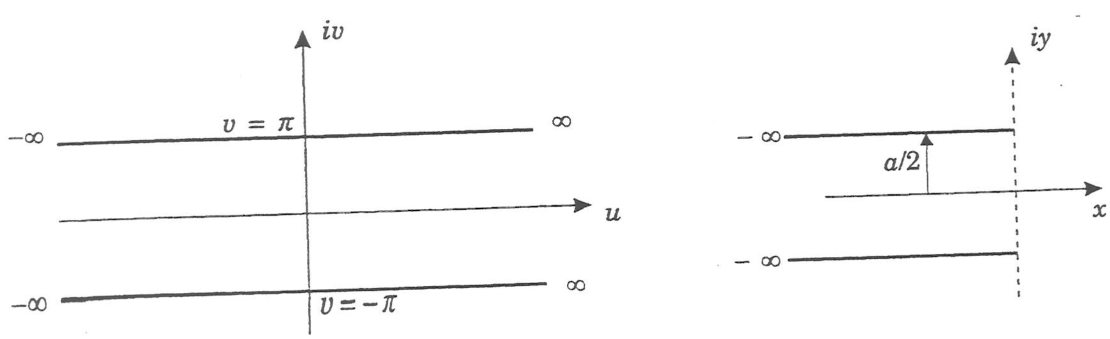


Figure 5.9 The infinite "planes" at $\pm i\pi$ in the f plane map to the semi-infinite "planes" in the z plane at $y = \pm a/2$ on the right.

capacitor. As we will see in the next section, we will also be able find the correction to the capacitance due to the fringing fields.

Letting the plates of the infinite capacitor in the f plane be at potential $\pm V_0/2$, with the top plate positive, the potential at any point in between the plates becomes $V = V_0 v/2\pi$. The analytic function of which V is the imaginary part is easily obtained as $\Phi(f) = V_0 f/2\pi$. Again it is somewhat awkward to substitute for f in terms of x and y. Instead, we separate the equation for z into real and imaginary components to get

$$x = \frac{a}{2\pi}\left(1 + u + e^{u}\cos v\right) \tag{5-18}$$

and

$$y = \frac{a}{2\pi}\left(v + e^{u}\sin v\right) \tag{5-19}$$

which we can evaluate parametrically by varying u to obtain the constant v (also constant $V = V_0 v/2\pi$) curves. Similarly, the electric field lines are obtained in the x-y plane by varying v. The field and potential lines are plotted in Figure 5.10.

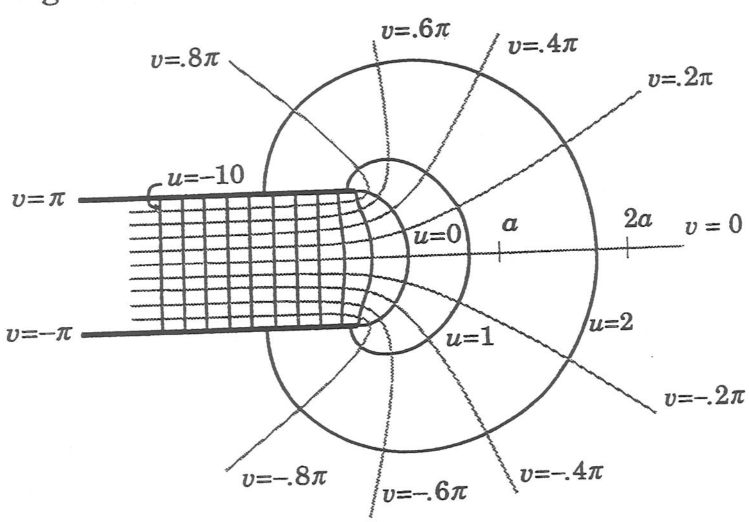


Figure 5.10 The equipotential and field lines near the edge of a semi-infinite plate capacitor are plotted parametrically.

Verifying that a given mapping does indeed map a particular surface onto another is fairly straightforward. It is not, however, obvious how mappings for given boundaries are to be obtained, other than perhaps by trial and error. For polygonal boundaries, a general method of constructing the mapping is offered by *Schwartz-Christoffel* transformations. For nonpolygonal boundaries, one must rely on "dictionaries" of mappings.†

† For both *Schwartz-Christoffel* transformations and a dictionary of mappings, see for instance K.J. Binns and P.J. Lawrenson. (1973) *Analysis and Computation of Electric and Magnetic Field Problems, 2nd ed.* Pergamon Press, New York.