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$$\Phi(\vec{r}) = \frac{kq}{r}\left[\frac{d}{r}\cos\theta + O\left(\frac{d^3}{r^3}\right)\right]$$

dipole term

no monopole term this time

Suppose we want all but the dipole term to vanish. This will be a "point dipole."

Take the limits; $q \to \infty$, $d \to 0$ such that

$qd \equiv \mu$ remains fixed.

$$\Phi_{\text{dipole}}(\vec{r}) = \frac{k\mu\cos\theta}{r^2} = \frac{k\vec{\mu}\cdot\vec{r}}{r^3}$$

The next term is proportional to

$$k\frac{q}{r}\frac{d^3}{r^3} = \frac{k(qd)d^2}{r^4} = \frac{k\mu d^2}{r^4} \to 0 \quad \text{as } d \to 0$$

$\vec{\mu}$ is the <u>dipole moment vector</u> RESERVE

$\vec{\mu}$ points from (−) to (+) charge.

Other multipoles:

monopole, dipole, quadrupole, octupole, ... 2^n-pole
$n=0$ $n=1$ $n=2$ $n=3$

You can now solve problem #3.

Mechanical Analogues:

Monopole – total charge $\Leftrightarrow$ total Mass of system

Dipole $\Leftrightarrow$ Center of Mass vector

Quadrupole $\Leftrightarrow$ Moment of Inertia Tensor

If you knew <u>all</u> of the multipole moments, you could reconstruct the charge distribution exactly.

Let's generalize the results for our specific examples. We seek a multipole expansion for the potential $\Phi(\vec{r})$.

Each term is composed of two factors:

① "Field Factor" - depends only on the coordinates of the point at which the potential is calculated, the unprimed field coordinates.

(e.g. $\frac{\vec{r}}{r^3}$ is the field factor in the point dipole potential.)

② "Source Factor" - depends only on the distribution of charge in the source.

(e.g. $\vec{\mu}$ is the source factor in the point dipole potential.)

For the general analysis, consider a charge distribution that is localized

$$\rho(\vec{r}\,') = 0 \quad \text{for } |\vec{r}\,'| > R$$

We will calculate $\Phi(\vec{r})$ only for points outside a sphere of radius R.

$$\underset{\text{Field}}{|\vec{r}|} > R > \underset{\text{source}}{|\vec{r}\,'|}$$

We will expand $\dfrac{1}{|\vec{r}-\vec{r}\,'|}$, which occurs in the expression for the potential $\Phi(\vec{r})$, in a 3-dimensional Taylor series.

Reminder: In 1-dimension, if $\underline{f}$ is differentiable to all orders (analytic), then

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \ldots$$

But this is not the Taylor series we will use. We want the "increment" form:

$$f(x+a) = f(x) + a f'(x) + \frac{1}{2!} a^2 f''(x) + \cdots$$

Now consider a scalar function of 3-d coordinates:

$$f(\vec{r}+\vec{a}) = f(\vec{r}) + \vec{a}\cdot\vec{\nabla} f(\vec{r}) + \frac{1}{2!}(\vec{a}\cdot\vec{\nabla})^2 f(\vec{r}) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i_1 \cdots i_n = 1}^{3} a_{i_1} \cdots a_{i_n} \frac{\partial^n (f(\vec{r}))}{\partial x_{i_1} \cdots \partial x_{i_n}}$$

where $\vec{a}\cdot\vec{\nabla} = \sum_{i_1=1}^{3} a_{i_1} \frac{\partial}{\partial x_{i_1}}$

and $(\vec{a}\cdot\vec{\nabla})^2 = \sum_{i_1, i_2 = 1}^{3} a_{i_1} a_{i_2} \frac{\partial^2}{\partial x_{i_1} \partial x_{i_2}}$, etc.

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for the multipole expansion:

$\vec{a} = -\vec{r}\,'$ and $f(\vec{r}-\vec{r}\,') = \frac{1}{|\vec{r}-\vec{r}\,'|}$

The Taylor series converges for $|\vec{r}| > |\vec{r}'|$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1 \cdots i_n = 1}^{3} x'_{i_1} \cdots x'_{i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left(\frac{1}{r}\right)$$

So

$$\Phi(\vec{r}) = \int_V dV' \frac{k\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1 \cdots i_n = 1}^{3} \bar{Q}_{i_1 \cdots i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left(\frac{1}{r}\right)$$

where $\bar{Q}_{i_1 \cdots i_n} = \int_V dV' \rho(\vec{r}') x'_{i_1} \cdots x'_{i_n}$

This is the 2^n-pole moment tensor

It is a tensor of rank $\underline{n}$ and by its definition can be seen to be symmetric under the interchange of <u>any</u> two indices.

Some examples to clarify the notation!

$\underline{n=0}$ $\bar{Q} = \int_V dV' \rho(\vec{r}\,') =$ monopole moment (total charge) scalar

$\underline{n=1}$ $\bar{Q}_i = \int_V dV' \rho(\vec{r}\,') x_i' =$ dipole moment vector

(this is what we called $\vec{\mu}$ previously)

$\underline{n=2}$ $\bar{Q}_{ij} = \int_V dV' \rho(\vec{r}\,') x_i' x_j' =$ quadrupole moment tensor

$\underline{n=3}$ $\bar{Q}_{ijk} = \int_V dV' \rho(\vec{r}\,') x_i' x_j' x_k' =$ octupole moment tensor

Notice that only the primed source coordinates $(\vec{r}\,')$ appear in $\bar{Q}$.

You are ready for problem #4

Warning ! The definition of multipole tensors varies from author to author. The definition of $\bar{Q}_{i_1 \dots i_n}$ arises in a natural way — through a Taylor series expansion in Cartesian coordinates. Later, we will expand in spherical coordinates and these are the multipole moments used by Jackson.

Why the bar over the $\bar{Q}_{i_1 \dots i_n}$?

It is customary (but not necessary) to redefine the Cartesian multipole moment tensors.

Right now, we have

$$\Phi(\vec{r}) = k\left(\frac{\bar{Q}}{r} - \vec{Q}\cdot\vec{\nabla}\left(\frac{1}{r}\right) + \frac{1}{2!}\sum_{i,j}\bar{Q}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) - + \cdots\right)$$

Note: the 2^n-pole term falls off as $\frac{1}{r^{n+1}}$. This is what makes the expansion useful, especially at large r.

In problem #2 we saw that

$$\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

we can similarly show that

$$\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) = \frac{1}{r^5}\left(3x_i x_j - r^2\delta_{ij}\right)$$

So we can write the potential as

$$\Phi(\vec{r}) = k\frac{\overline{Q}}{r} + k\frac{\vec{\overline{Q}}\cdot\vec{r}}{r^3} + \frac{1}{2!}\sum_{ij}\overline{Q}_{ij}\frac{1}{r^5}\left(3x_i x_j - r^2\delta_{ij}\right)$$

Consider the tensor: $3x_i x_j - r^2\delta_{ij}$

Its trace is $\sum_{ij}\left(3x_i x_j - r^2\delta_{ij}\right)\delta_{ij}$

or $\sum_i\left(3x_i x_i - r^2\delta_{ii}\right) = \left(3r^2 - 3r^2\right) = 0$

Therefore, we may <u>add</u> to $\overline{Q}_{ij}$ any multiple of the unit tensor δ_{ij} without changing the electrostatic potential!

Define: $Q_{ij} = \overline{Q}_{ij} + A\,\delta_{ij}$

where A is arbitrary.

The new quadrupole piece of $\Phi(\vec{r})$ is

$$\frac{1}{2}\sum_{ij}(3x_i x_j - r^2\delta_{ij})Q_{ij} = \frac{1}{2}\sum_{ij}(3x_i x_j - r^2\delta_{ij})(\overline{Q}_{ij} + A\delta_{ij})$$

$$= \frac{1}{2}\sum_{ij}(3x_i x_j - r^2\delta_{ij})\,\overline{Q}_{ij} + A(0)$$

same as the old quadrupole piece.

It is customary to choose $A = -\frac{1}{3}\sum_{k}\overline{Q}_{kk}$

that is $\left(-\frac{1}{3}\right)$ times the trace of the old tensor.

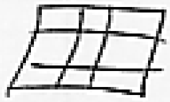
$$Q_{ij} = \overline{Q}_{ij} - \frac{1}{3}\Big(\sum_{k}\overline{Q}_{kk}\Big)\delta_{ij}$$

Why do this?

Because Q_{ij} is now traceless.

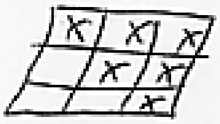
$$Tr[Q_{ij}] = \sum_i Q_{ii} = \sum_i \left(\bar{Q}_{ii} - \frac{1}{3} \sum_k \bar{Q}_{kk} \delta_{ii} \right)$$

$$= \sum_i \bar{Q}_{ii} - \frac{1}{3}(3) \sum_k \bar{Q}_{kk} = 0$$

In 3 dimensions, a 2nd rank tensor (matrix) has 9 components 

general — 9 independent elements

symmetric — 6 independent elements



symmetric + traceless — 5 independent elements since $Q_{11} + Q_{22} + Q_{33} = 0$

A traceless, symmetric rank n tensor has $2n+1$ independent components.

$Q_{i_1 \cdots i_n}$ are called "irreducible" tensors because all of their elements are independent.

Translation Dictionary:

$Q = \bar{Q}$ no change in the total charge,

$Q_i = \bar{Q}_i$ no change in the dipole moment vector.

$$Q_{ij} = \bar{Q}_{ij} - \frac{1}{3}\left(\sum_k Q_{kk}\right)\delta_{ij}$$

the connection between $Q_{i_1 \cdots i_n}$ and $\bar{Q}_{i_1 \cdots i_n}$ is more complicated for $n > 2$.

$$\Phi(\vec{r}) = k \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{i_1 \cdots i_n = 1}^{3} Q_{i_1 \cdots i_n} \frac{\partial^n}{\partial x_{i_1} \cdots \partial x_{i_n}} \left(\frac{1}{r}\right)$$

is the <u>same</u> potential as before.

The spherical tensors used by Jackson which we will meet later are automatically <u>irreducible</u>. These are also used extensively in Quantum Mechanics.

End Lecture #2 <u>RESERVE</u>