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We are now in a position to find what is needed to specify the electrostatic potential in a finite region of space V, if the charge density is known only in that region.

We will require two mathematical relations called Green's Identities, both of which can be derived from the

Divergence Theorem: $\int_V dV\, \vec{\nabla}\cdot\vec{u}(\vec{r}) = \oint_S dS\, \hat{n}\cdot\vec{u}(\vec{r})$

where the closed surface S bounds the volume V.

First take $\vec{u}(\vec{r}) = \psi(\vec{r})\left(\vec{\nabla}\psi(\vec{r})\right)$

where $\psi(\vec{r})$ is some scalar field.

Green's First Identity follows:

$$\int_V dV\left[\psi\nabla^2\psi + (\vec{\nabla}\psi)\cdot(\vec{\nabla}\psi)\right] = \oint_S dS\, \psi\hat{n}\cdot(\vec{\nabla}\psi)$$

RESERVE

Green's Second Identity is derived by taking

$\vec{u} = \psi \vec{\nabla} \Phi - \Phi \vec{\nabla} \psi$ in the divergence theorem.

$$\int_V dV \left[\psi \nabla^2 \Phi - \Phi \nabla^2 \psi \right] = \oint_S dS \left[\psi \hat{n} \cdot (\vec{\nabla} \Phi) - \Phi \hat{n} \cdot (\vec{\nabla} \psi) \right]$$

We will consider the second identity with the following substitutions:

$\vec{r}'$ is going to be the (dummy) variable of integration

$\Phi(\vec{r}')$ will be the electrostatic potential

$\psi(\vec{r}') \equiv \frac{1}{|\vec{r} - \vec{r}'|}$ is a scalar function

and

$\vec{r}$ will be a (fixed) observation point somewhere <u>inside</u> the volume V.

Green's Second Identity gives:

$$\int_V dV' \left[\frac{1}{|\vec{r}-\vec{r}'|} \nabla'^2 \Phi(\vec{r}') - \Phi(\vec{r}') \nabla'^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) \right]$$

$$= \oint_S dS' \left[\frac{1}{|\vec{r}-\vec{r}'|} \hat{n}' \cdot \vec{\nabla}' \Phi(\vec{r}') - \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}' \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) \right]$$

Notice:

$$\nabla'^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi \delta^3(\vec{r}-\vec{r}')$$ as shown in the last lecture.

$$\nabla'^2 \Phi(\vec{r}') = -4\pi \rho(\vec{r}')$$ Poisson's Equation

$$\vec{\nabla}' \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$ Problem # 2

So we have

$$\Phi(\vec{r}) = \int_V dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S dS' \frac{\hat{n}' \cdot \vec{\nabla}' \Phi(\vec{r}') \frac{1}{4\pi}}{|\vec{r}-\vec{r}'|}$$

$$- \oint_S dS' \frac{\Phi(\vec{r}') \hat{n}' \cdot (\vec{r}-\vec{r}') \frac{1}{4\pi}}{|\vec{r}-\vec{r}'|^3}$$

The three terms on the right have the following physical interpretation:

The first term $\int_V dV' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$ is the potential due to the distribution of charges <u>within</u> V.

No surprises here, we have seen this before.

The second and third terms must be due to charges <u>outside</u> V about which we have no knowledge.

The second term $\oint_S dS' \frac{\hat{n}' \cdot \vec{\nabla}_{r'} \Phi(\vec{r}') \frac{1}{4\pi}}{|\vec{r}-\vec{r}|}$

looks like the potential due to a surface charge density (monopole density)

$$\int_S dS' \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \text{where } \sigma(\vec{r}') = \frac{1}{4\pi} \hat{n}' \cdot \vec{\nabla}_{r'} \Phi(\vec{r}')$$

is the "effective" surface charge density on the completely artificial surface S.

(S is a mathematical construct, not a physical surface. You are free to choose S as you please.)

RESERVE

The third term $-\oint_S dS' \frac{\Phi(\vec{r}')\hat{n}'\cdot(\vec{r}-\vec{r}')\frac{1}{4\pi}}{|\vec{r}-\vec{r}'|^3}$

looks like the potential due to a surface distribution of <u>dipoles</u>

$$\int_S dS' \frac{\vec{D}(\vec{r}')\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} .$$

where $\vec{D}(\vec{r}') = \frac{-1}{4\pi}\hat{n}'\Phi(\vec{r}')$ is the "effective" surface dipole moment density.

You may wonder how $\Phi(\vec{r}')$ can appear in the two surface integrals when we are trying to find $\Phi(\vec{r})$ inside V.

The answer to this is that we must specify $\Phi(\vec{r}')$ with $\vec{r}'$ on the boundary S, <u>or</u> we must specify $\vec{\nabla}_{r'}\Phi(\vec{r}')$ again on the boundary S. After these specifications, $\Phi(\vec{r})$ is uniquely determined inside the volume V.

RESERVE

Unfortunately, we cannot specify both $\Phi(\vec{r}')$ and $\vec{\nabla}_{r'}\Phi(\vec{r}')$ on S. Either one alone is sufficient to determine $\Phi(\vec{r})$ inside V. Specifying both would overdetermine the problem.

Uniqueness of the Solution

Suppose that we have two solutions for the potential within V. Call them $\Phi_1(\vec{r})$ and $\Phi_2(\vec{r})$. They both must satisfy the Poisson Equation

$$\left.\begin{aligned} \nabla^2\Phi_1(\vec{r}) &= -4\pi\rho(\vec{r}) \\ \nabla^2\Phi_2(\vec{r}) &= -4\pi\rho(\vec{r}) \end{aligned}\right\} \text{ for } \vec{r} \text{ inside } V$$

Imagine that the mathematical surface S is composed of two pieces, S_A and S_B.

Let $\Phi_1(\vec{r}') = \Phi_2(\vec{r}')$ on S_A

and $\hat{n}'\cdot\vec{\nabla}_{r'}\Phi_1(\vec{r}') = \hat{n}'\cdot\vec{\nabla}_{r'}\Phi_2(\vec{r}')$ on S_B

The usual uniqueness argument proceeds as follows:

Define $\psi(\vec{r}) \equiv \Phi_1(\vec{r}) - \Phi_2(\vec{r})$. Then

$$\nabla^2 \psi = \nabla^2(\Phi_1 - \Phi_2) = 0 \quad \text{in } V$$

(so ψ satisfied Laplace's Equation)

and $\psi = 0$ on S_A

$$\hat{n} \cdot \vec{\nabla} \psi = 0 \quad \text{on } S_B$$

Now we will employ Green's First Identity

$$\int_V dV \left[\psi \nabla^2 \psi + (\vec{\nabla}\psi) \cdot (\vec{\nabla}\psi) \right] = \oint_{S_A + S_B} dS \; \psi \, \hat{n} \cdot (\vec{\nabla}\psi)$$

$$= \int_{S_A} dS \; \psi \, \hat{n} \cdot (\vec{\nabla}\psi) + \int_{S_B} dS \; \psi \, \hat{n} \cdot (\vec{\nabla}\psi)$$

So we are left with

$$\int_V dV \; (\vec{\nabla}\psi)^2 = 0$$

But the integrand is a squared vector, a positive semi-definite quantity. The only way for the integral to vanish is if $|\vec{\nabla}\psi| = 0$

This implies that $\vec{\nabla}\psi = 0$ and therefore ψ = constant everywhere in V

If S_A is not null, then since $\psi = 0$ on S_A $\psi = 0$ everywhere within V.

If S_A is null, that is $S = S_B$, then the argument stops at ψ = constant in V.

If $\psi = 0$ (there is some S_A) then

$\Phi_1(\vec{r}) = \Phi_2(\vec{r})$ everywhere in V and the solution is <u>unique</u>.

If ψ = constant (the whole surface is S_B) then

$\Phi_1(\vec{r}) = \Phi_2(\vec{r})$ + constant in V.

The solutions are unique <u>up to a constant term</u>.

RESERVE

Boundary Conditions:

Specify $\Phi(\vec{r}')$ on S_A $\Leftrightarrow$ Dirichlet B.C.

Specify $\vec{\nabla}_{r'}\Phi(\vec{r}')$ on S_B $\Leftrightarrow$ Neumann B.C.

Let's tackle the simplest case first.

Let us solve for $\Phi(\vec{r})$ inside V when $\Phi(\vec{r}')$ is specified on all of S. $(S = S_A)$

This is the Dirichlet problem.

We begin with the Second Green Identity:

$$\int_V dV' \left[\psi(\vec{r}')\nabla_{r'}^2\Phi(\vec{r}') - \Phi(\vec{r}')\nabla_{r'}^2\psi(\vec{r}')\right]$$

$$= \oint_S dS' \left[\psi(\vec{r}')\hat{n}'\cdot\vec{\nabla}_{r'}\Phi(\vec{r}') - \Phi(\vec{r}')\hat{n}'\cdot\vec{\nabla}_{r'}\psi(\vec{r}')\right]$$

and we will demand that $\psi(\vec{r}')$ is such that

$$\nabla_{r'}^2\psi(\vec{r}') = -4\pi\delta^3(\vec{r}-\vec{r}') \quad \text{for } \vec{r}, \vec{r}' \text{ in } V$$

and $\psi(\vec{r}') = 0$ for $\vec{r}'$ on S.

RESERVE

We won't actually find such a function yet, but simply define the properties we want.

This kind of function is called the Dirichlet Green function for the geometry described by S.

$$\psi(\vec{r}') \equiv G_D(\vec{r}, \vec{r}')$$

So explicitly

$$\nabla'^2 G_D(\vec{r}, \vec{r}') = -4\pi \delta^3(\vec{r} - \vec{r}')$$

and

$$G_D(\vec{r}, \vec{r}') = 0 \quad \text{for } \vec{r}' \text{ on } S$$

Green's Second Identity becomes

$$\Phi(\vec{r}) = \int_V dV' G_D(\vec{r}, \vec{r}') \rho(\vec{r}') - \oint_S dS' \frac{1}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}'_{r'} G_D(\vec{r}, \vec{r}')$$

This is the potential inside V, exactly what we want, given $\Phi(\vec{r}')$ on S (Dirichlet boundary condition)

RESERVE

The Dirichlet Green function $G_D(\vec{r}, \vec{r}')$ is a remarkable function!

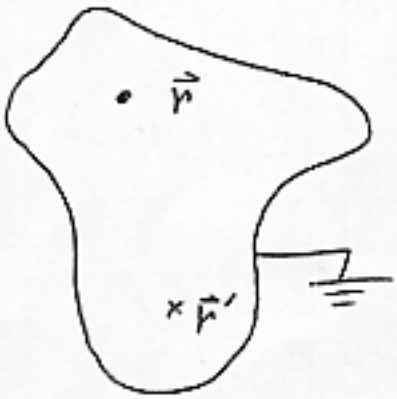
Imagine placing a unit ($q=1$) point charge in a conductor at point $\vec{r}'$. Now ground (earth) the conductor so it is sitting at potential $\Phi = 0$.

$G_D(\vec{r}, \vec{r}')$ is the potential at the point $\vec{r}$!

Another amazing property of $G_D(\vec{r}, \vec{r}')$ is that it is symmetric;

$$G_D(\vec{r}, \vec{r}') = G_D(\vec{r}', \vec{r})$$

Physically, this is even more incredible.



Place the unit point charge at $\vec{r}$ this time and measure the potential at $\vec{r}'$, with the surface grounded. The potential will be the same as above!

End Lecture # 4