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Last time, we claimed that the Dirichlet Green function was symmetric under the interchange of its arguments:

$$G_D(\vec{r}_1, \vec{r}_2) = G_D(\vec{r}_2, \vec{r}_1)$$

The proof goes by Green's Second Identity with $\psi(\vec{r}\,') = G_D(\vec{r}_1, \vec{r}\,')$ and $\Phi(\vec{r}\,') = G_D(\vec{r}_2, \vec{r}\,')$

then

$$\int_V dV' \Big[G_D(\vec{r}_1, \vec{r}\,') \underbrace{\nabla^2_{r'} G_D(\vec{r}_2, \vec{r}\,')}_{-4\pi\,\delta^3(\vec{r}_2, \vec{r}\,')} - G_D(\vec{r}_2, \vec{r}\,') \underbrace{\nabla^2_{r'} G_D(\vec{r}_1, \vec{r}\,')}_{-4\pi\,\delta^3(\vec{r}_1, \vec{r}\,')} \Big] = 0$$

the surface integral in Green's Second Identity vanishes because both $\psi(\vec{r}\,')$ and $\Phi(\vec{r}\,')$ vanish on S. This is one of the defining properties of the Dirichlet Green function.

the delta functions above make the integral easy.

We get $-4\pi \left[G_D(\vec{r}_1, \vec{r}_2) - G_D(\vec{r}_2, \vec{r}_1) \right] = 0$

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Reminder and Comparison:

Dirichlet Boundary Conditions	Neumann Boundary Conditions
$\Phi(\vec{r}')$ specified on S	$\hat{n}' \cdot \vec{\nabla}_{r'} \Phi(\vec{r}')$ specified on S (normal component of $\vec{E}$)
$\nabla^2_{r'} G_D(\vec{r},\vec{r}') = -4\pi \delta^3(\vec{r}-\vec{r}')$ for $\vec{r}, \vec{r}'$ in V	$\nabla^2_{r'} G_N(\vec{r},\vec{r}') = -4\pi \delta^3(\vec{r},\vec{r}')$ for $\vec{r}, \vec{r}'$ in V
$G_D(\vec{r},\vec{r}') = 0$ for $\vec{r}'$ on S	?

It is tempting to fill in the $\boxed{?}$ above with an analogous condition like

$$\hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}') = 0 \quad \text{for } \vec{r}' \text{ on } S,$$

but this leads to a contradiction.

We can see this by using the divergence theorem:

$$\oint_S dS' \, \underbrace{\hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}')}_{\to 0} = \int_V dV' \, \nabla^2_{r'} G_N(\vec{r},\vec{r}')$$

$$\text{(suppose this is 0)} = \int_V dV' \left[-4\pi \delta^3(\vec{r}-\vec{r}')\right]$$

$$0 \neq -4\pi$$

If $\hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}')$ cannot be set to 0 on the bounding surface S, try the next simplest choice — a constant.

$$\boxed{\hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}') = -\frac{4\pi}{S}} \quad \text{for } \vec{r}' \text{ on } S$$

where "S" is the surface area of the boundary.

Let's see if this works:

$$\oint_S dS' \, \hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}') = \int_V dV' \, \nabla^2_{r'} G_N(\vec{r}-\vec{r}')$$

$$\left(-\frac{4\pi}{S}\right) \oint_S dS' = \left(-\frac{4\pi}{S}\right) S = -4\pi \quad \checkmark$$

We can now use $G_N(\vec{r},\vec{r}')$ to find the potential inside the volume V knowing only $\hat{n}' \cdot \vec{\nabla}_{r'} \Phi(\vec{r}')$ on the surface S.

$$\Phi(\vec{r}) = \int_V dV' \rho(\vec{r}') G_N(\vec{r},\vec{r}') + \oint_S \frac{dS'}{4\pi} \hat{n}' \cdot \vec{\nabla}_{r'} \Phi(\vec{r}') G_N(\vec{r},\vec{r}')$$

$$- \oint_S \frac{dS'}{4\pi} \Phi(\vec{r}') \hat{n}' \cdot \vec{\nabla}_{r'} G_N(\vec{r},\vec{r}')$$

The last term (which vanished for the Dirichlet case) can be simplified:

$$-\oint_S \frac{dS'}{4\pi}\, \Phi(\vec{r}\,')\, \hat{n}'\cdot\vec{\nabla}_{r'}\, G_N(\vec{r},\vec{r}\,') = -\oint_S \frac{dS'}{4\pi}\, \Phi(\vec{r}\,')\left(-\frac{4\pi}{S}\right)$$

$$= \frac{1}{S}\oint_S dS'\, \Phi(\vec{r}\,') \equiv \langle \Phi \rangle_S$$

this is the <u>average</u> potential over the surface S.

So...

(from known charges inside V)

$$\Phi(\vec{r}) = \langle \Phi \rangle_S + \int_V dV'\, \rho(\vec{r}\,')\, G_N(\vec{r},\vec{r}\,')$$

$$+ \oint_S \frac{dS'}{4\pi}\, \hat{n}'\cdot\vec{\nabla}_{r'}\Phi(\vec{r}\,')\, G_N(\vec{r},\vec{r}\,')$$

(from boundary conditions)

for $\vec{r}$ inside V.

c) Energy in the Electro-Magnetic Field

We will derive our results through the work-energy theorem, which involves forces acting through distances, but at the end of the day we will have an expression for energy in terms of the electric field.

Consider a collection of point charges q_i located at positions $\vec{r}_i$.

The force on the ith charge due to all the other charges is:

$$\vec{F}_i = \sum_{\substack{j \\ (j \neq i)}} \frac{q_i q_j (\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^3} = - \sum_{\substack{j \\ (j \neq i)}} q_i q_j \vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

where we have used a familiar trick in the last step.

The work done <u>by</u> some external agent <u>on</u> the charges in moving from some initial configuration $\{\vec{r}_i^{\,A}\}$ to some final configuration $\{\vec{r}_i^{\,B}\}$ is:

$$W_{A\to B} = \sum_i \int_{\vec{r}_i^{\,A}}^{\vec{r}_i^{\,B}} \vec{F}_i \cdot d\vec{r}_i = -\sum_i \sum_{\substack{j \\ (j\neq i)}} q_i q_j \int_A^B d\vec{r}_i \cdot \vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

$$= -\frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} q_i q_j \int_A^B \left(d\vec{r}_i \cdot \vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_j|} + d\vec{r}_j \cdot \vec{\nabla}_j \frac{1}{|\vec{r}_i - \vec{r}_j|} \right)$$

since the sum is symmetric in <u>i</u> and <u>j</u>.

Now some definitions:

$$\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j \qquad \vec{\nabla}_{ij} \equiv \vec{\nabla}_{r_{ij}}$$

$$d\vec{r}_{ij} = d\vec{r}_i - d\vec{r}_j$$

Notice that:

$$\vec{\nabla}_j \frac{1}{|\vec{r}_i - \vec{r}_j|} = -\vec{\nabla}_i \frac{1}{|\vec{r}_i - \vec{r}_j|} = -\vec{\nabla}_{ij} \frac{1}{|\vec{r}_i - \vec{r}_j|} \left(= \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \right)$$

the work done can then be written as:

$$W_{A\to B} = -\frac{1}{2} \sum_{\substack{i,j \\ (i\neq j)}} q_i q_j \int_A^B d\vec{r}_{ij} \cdot \vec{\nabla}_{ij} \frac{1}{|\vec{r}_i - \vec{r}_j|}$$

but $\sum_k d\vec{r}_k \cdot \vec{\nabla}_k$ is just a <u>total</u> <u>derivative</u>.

$$W_{A\to B} = -\frac{1}{2} \sum_{\substack{i,j \\ (i\neq j)}} q_i q_j \int_A^B d\left(\frac{1}{|\vec{r}_i - \vec{r}_j|}\right)$$

and the integral of a total derivative is completely trivial.

Now remember from elementary mechanics that the Δpotential energy is the negative of the work done.

$$W_{A\to B} \equiv -\Delta U = -(U_B - U_A)$$

The electrostatic potential energy for point charges q_i at positions $\vec{r}_i$ is:

$$U = \frac{1}{2} \sum_{\substack{i,j \\ (i\neq j)}} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

And we recognize this form as the sum of the charges times the electrostatic potentials due to all the other charges.

If the charge is distributed continuously, we make the following substitutions:

$$\vec{r}_i \to \vec{r} \qquad q_i \to dV \rho(\vec{r})$$
$$\vec{r}_j \to \vec{r}\,' \qquad q_j \to dV' \rho(\vec{r}\,')$$

to get

$$U = \frac{1}{2} \int dV \int dV' \frac{\rho(\vec{r}) \rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

$$= \frac{1}{2} \int dV \rho(\vec{r}) \Phi(\vec{r}) \qquad \text{where } \Phi(\vec{r}) = \int dV' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}$$

Notice that both $\vec{r}$ and $\vec{r}\,'$ are dummy variables above. After the double volume integrals, there is no spacial dependence left in U. The electrostatiz potential energy is not a function of position, rather it is a universal <u>number</u>.

In the sums, we carefully avoided setting $i=j$, but in the continuous form we let the integration variables overlap. That is, we made no restriction like $\vec{r} \neq \vec{r}\,'$. This is ok because the error that we make is proportional to dV, and this is <u>infinitesimal</u>.

However, if the charge is not spread out we encounter a problem. We learned how to change back to the discrete case from the continuous — just use

$$\rho(\vec{r}) = \sum_i q_i \, \delta^3(\vec{r}-\vec{r}_i)$$

with this substitution,

$$U = \frac{1}{2} \underbrace{\sum_{\substack{i,j \\ (i \neq j)}} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}}_{\text{recover previous result}} + \underbrace{\cancel{\frac{1}{2} \sum_i \frac{q_i^2}{0}}^{\to\infty}}_{\text{new infinite "self energies"}}$$

Problem #9 explores this issue

Now we can write U in terms of the electric field $\vec{E}$.

$$U = \frac{1}{2} \int dV \, \rho(\vec{r}) \, \Phi(\vec{r})$$

Use Poisson's Equation: $\rho(\vec{r}) = \frac{1}{4\pi} \vec{\nabla} \cdot \vec{E}$

$$U = \frac{1}{8\pi} \int_V dV \, (\vec{\nabla} \cdot \vec{E}(\vec{r})) \, \Phi(\vec{r})$$ integrate by parts

$$= \frac{1}{8\pi} \int_V dV \, \vec{\nabla} \cdot (\vec{E} \Phi) - \frac{1}{8\pi} \int_V dV \, \vec{E} \cdot (\vec{\nabla} \Phi)$$

$\boxed{\vec{E} = -\vec{\nabla} \Phi}$

$$= \frac{1}{8\pi} \oint_S dS \, \hat{n} \cdot \vec{E} \Phi + \frac{1}{8\pi} \int_V dV \, E^2$$

If the charges are distributed out to infinity then the potential energy will be infinite. This is not interesting, so...

Assume $\rho(\vec{r})$ vanishes outside a sphere of radius R, then as $R \to \infty$

$\Phi(\vec{R}) \sim$ at <u>least</u> as fast as $\frac{1}{R}$

remember the multipole expansion starts at $1/R$ in general and if $Q=0$ it starts at $1/R^2$, and so on.

$\hat{n} \cdot \vec{E}(\vec{R}) \sim$ at least as fast as $\frac{1}{R^2}$

$dS \sim R^2 d\Omega$

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so $dS\, \hat{n} \cdot \vec{E}\, \Phi \sim \frac{1}{R} \to 0$ as $R \to \infty$

Now we can set the surface term to zero:

$$U = \frac{1}{8\pi} \int dV\, E^2 \quad \text{or} \quad U = \int dV\, u(\vec{r})$$

where $u(\vec{r}) = \dfrac{\vec{E}^2(\vec{r})}{8\pi}$ is the <u>energy density</u>.

— End Lecture #5 —